Math 3110: Linear Algebra (Spring 2014)
Midterm 1
February 25, 2014

NAME:

Instructions

- Exam time is 75 minutes
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

Question	Points Earned	Maximum Points
1		8
2		4
3		6
4		11
5		10
6		5
7		6
Total		**50**

Student to your left:

Student to your right:

1. Consider the 3×4 matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. For each of "the four subspaces" associated with $\boldsymbol{A}$, compute its dimension and then find a basis for it. [*Hint*: You should be able to do all of this purely by inspection.]

   (a) The column space, $C(\boldsymbol{A})$.

   (b) The row space, $C(\boldsymbol{A}^T)$.

   (c) The nullspace, $N(\boldsymbol{A})$.

   (d) The left nullspace, $N(\boldsymbol{A}^T)$.

2. Compute the inverse of the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

3. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix (m rows, and n columns) of rank r. Additionally, suppose that $\boldsymbol{Ax} = \boldsymbol{b}$ has a *unique* solution.

   (a) List all inequalities that must hold between n, m, and r.

   (b) For each of the four fundamental subspaces of $\boldsymbol{A}$, say as much as you can about what its dimension could be.

   (c) The equation $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{c}$ (always)(not always) has (a unique solution)(many solutions)(no solution).

4. For each statement below, circle the option that makes it correct.

   (a) If $n < m$, then the column vectors of an $m \times n$ matrix (do)(do not)(might not) span $\boldsymbol{R}^m$.

   (b) If $n < m$, then the column vectors of an $m \times n$ matrix (do)(do not)(might not) span $C(\boldsymbol{A})$.

   (c) If $n > m$, then the column vectors of an $m \times n$ matrix (do)(do not)(might not) span $\boldsymbol{R}^m$.

   (d) If $n > m$, then the column vectors of an $m \times n$ matrix (do)(do not)(might not) span $C(\boldsymbol{A})$.

   Now, suppose S and T are subspaces of $\mathbb{R}^{10}$ with $\dim S = 6$ and $\dim T = 7$. Answer the following questions.

   (i) The smallest that $\dim(S+T)$ could be is ______________________.

   (ii) The largest that $\dim(S+T)$ could be is ______________________.

   (iii) The smallest that $\dim(S \cap T)$ could be is ______________________.

   (iv) The largest that $\dim(S \cap T)$ could be is ______________________.

   (v) The union

   $$S \cup T = \{v \in \boldsymbol{R}^{10} \mid v \in S \text{ or } v \in T\}$$

   is a subspace of $\boldsymbol{R}^{10}$ if and only if ____________________________________.

   (vi) Suppose that S is the nullspace of an 8×10 matrix $\boldsymbol{A}$. Determine the dimensions of the other four fundamental subspaces of $\boldsymbol{A}$.

5. Let $\boldsymbol{A}$ be a 3×3 matrix, and suppose that $\boldsymbol{Ax} = \boldsymbol{b}$ has solution

$$\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

(a) What is the rank of $\boldsymbol{A}$?

(b) What are the dimensions of the four fundmental subspaces of $\boldsymbol{A}$?

(c) Suppose that using elimination to solve $\boldsymbol{Ax} = \boldsymbol{b}$ yields the equation $\boldsymbol{Rx} = \boldsymbol{d}$, where $\boldsymbol{R}$ is the row reduced echelon form of $\boldsymbol{A}$. (That is, every pivot is 1, and there are 0's above and below the pivots.) Find $\boldsymbol{R}$ and $\boldsymbol{d}$.

(d) Suppose that elimination was done by subtracting 3 times row 1 from row 2, and then 5 times row 1 from row 3. What is the "elimination matrix" $\boldsymbol{E}$ such that $\boldsymbol{EA} = \boldsymbol{R}$?

(e) Find $\boldsymbol{A}$ and $\boldsymbol{b}$.

6. Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$. Determine if $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$. If "yes," then write $\boldsymbol{b}$ as an explicit linear combination of the columns of $\boldsymbol{A}$. If "no," then find an explicit condition on the b_i's that a vector $\boldsymbol{b} = (b_1, b_2, b_3)$ in the column space must satisfy for $\boldsymbol{Ax} = \boldsymbol{b}$ to have a solution (that naturally, $\boldsymbol{b}$ does *not* satisfy).

7. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Find all solutions to the system $\boldsymbol{Ax} = \boldsymbol{0}$.

   (b) Suppose $\boldsymbol{b}$ is the sum of the columns of $\boldsymbol{A}$. Write down a complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$.