Math 3110: Linear Algebra (Spring 2014)
Midterm 2
April 1, 2014

NAME:

Instructions

- Exam time is 75 minutes
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

Question	Points Earned	Maximum Points
1		7
2		8
3		8
4		4
5		5
6		9
7		9
Total		**50**

Student to your left:

Student to your right:

1. (7 pts) Give an example of each of the following:

   (a) A non-identity permutation matrix with determinant 1.

   (b) A set S of vectors for which $(S^{\perp})^{\perp} \neq S$.

   (c) An *orthonormal basis* for $\mathbb{R}^3$, including the vector $\boldsymbol{q}_1 = (1,1,0)/\sqrt{2}$.

   (d) An orthogonal matrix $\boldsymbol{Q}$ such that $\det \boldsymbol{Q} \neq 1$.

   (e) A 3×3 projection matrix of rank 1.

   (f) A 3×3 projection matrix of rank 2.

   (g) A set of 3 distinct nonzero vectors in $\mathbb{R}^3$ for which the Gram-Schmidt process will definitely fail.

2. (8 pts) Fill in the blanks. No explanations needed, but think carefully before answering!

   (a) Let $\widehat{\boldsymbol{x}}$ be the least-squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$. Then $\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}$ lies in the

   __________________________ of $\boldsymbol{A}$.

   (b) When attempting to solve $\boldsymbol{Ax} = \boldsymbol{b}$, suppose $\boldsymbol{b}$ is in $N(\boldsymbol{A}^T)$. Then the least squares solution $\widehat{\boldsymbol{x}}$

   lies in ______________________________ (which subspace?).

   (c) The matrices $\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{A}$ always have the same ______________________________.

   (d) Suppose $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$, and $\boldsymbol{v} \in C(\boldsymbol{A})$, and $\boldsymbol{w} \in N(\boldsymbol{A}^T)$. Then

      (i) $\boldsymbol{Pv} =$ __________________________

      (ii) $\boldsymbol{Pw} =$ __________________________

   (e) The determinant of an $n \times n$ projection matrix $\boldsymbol{P}$ is non-zero if and only if $\boldsymbol{P}$ is the projection

   onto _____________________ (what subspace?).

   (f) As long as $\boldsymbol{A}$ is _____________________, the Gram-Schmidt process allows one to factor the

   matrix $\boldsymbol{A} = \boldsymbol{QR}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is _____________________.

3. (8 pts) Find the determinant of the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 8 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}.$$

[*Hint*: Cofactor expansion will work to find $\det \boldsymbol{A}$, but it is certainly not necessary.]

4. (4 pts) Suppose a 4×4 matrix $\boldsymbol{A}$ has $\det \boldsymbol{A} = -3$. Find:

   (a) $\det(\frac{1}{2}\boldsymbol{A})$

   (b) $\det(-\boldsymbol{A})$

   (c) $\det(\boldsymbol{A}^2)$

   (d) $\det(\boldsymbol{A}^{-1})$

5. (5 pts) Use the Gram-Schmidt process to turn $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ into an orthonormal set.

6. (9 pts) Let L be the line $x = y = z$ in $\mathbb{R}^3$.

   (a) Find a basis for $L^\perp$, the *orthogonal complement* of the line L

   (b) Project the vector $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the line L.

   (c) Project the vector $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto $L^\perp$.

   (d) Write the vector $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a sum: $\boldsymbol{b} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v} \in L$ and $\boldsymbol{w} \in L^\perp$.

7. (9 points) Consider the points $(0,0)$, $(1,2)$, and $(2,2)$ in $\mathbb{R}^2$.

   (a) Use the least sqaures method to find the best fit line $Ct+D$ through these three points. Recall that to do this, you will need to solve an equation of the form

   $$\boldsymbol{A}\begin{bmatrix} C \\ D \end{bmatrix} = \boldsymbol{b}\,.$$

   for some matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$.

   (b) For the $\boldsymbol{b}$ you found above, calculate the (Euclidean) distance from it to the column space of $\boldsymbol{A}$.