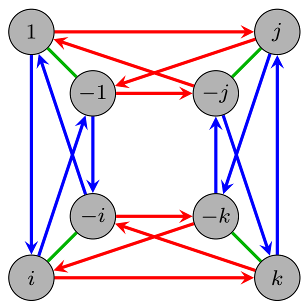
MthSc 412: Abstract Algebra (Fall 2010)
Midterm 2

NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (15 points) Consider the following Cayley diagram for Q_4.



Draw the subgroup lattice (Hasse diagram) for Q_4. Label each edge with the corresponding index.

2. (10 points)

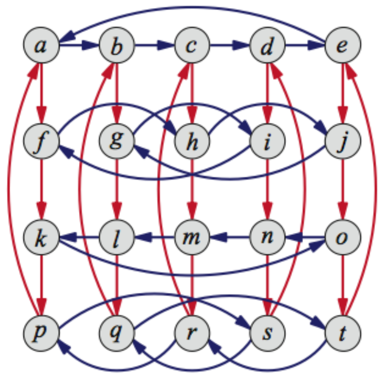
   (a) Exhibit an element of S_9 of order 20.

   (b) How many elements of order 18 are there in S_9?

3. (5 points) How many subgroups does C_{24} have?

4. (10 points) How many homomorphisms are there from C_{412} to C_{666}? Use the Fundamental Homomorphism Theorem to fully justify your answer.

5. (20 points) Let G be given by the following Cayley diagram, and assume that e is the identity. Also, all arrows running north-south are of one type (j) and all arrows running east-west are another type (a). Let $H = \{e, j, o, t\}$, and $K = \{a, b, c, d, e\}$. Compute the normalizers of the following subgroups: $\{e\}$, H, K, G.



6. (4 points each) Answer true or false to each statement. If false, give a counterexample. Assume that each $H_i \lhd G_i$ for $i = 1, 2$.

   (a) If every proper subgroup H of a group G is cyclic, then G is cyclic.

   (b) If $K \lhd H \lhd G$, then $K \lhd G$.

   (c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

   (d) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, then $H_1 \cong H_2$.

   (e) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

7. (20 points) Let $\phi\colon G \to H$ be a homomorphism, and let $K = \ker\phi$. In this problem we will prove the Fundamental Homomorphism Theorem: $G/K \cong \operatorname{Im}\phi$. Define the map

$$i\colon G/K \longrightarrow \operatorname{Im}\phi, \qquad i(gK) = \phi(g)\,.$$

(a) Show that i is *well-defined*, i.e., that if $gK = hK$, then $i(gK) = i(hK)$.

(b) Show that i is a homomorphism.

(c) Show that i is *injective* (1–1, i.e., that if $i(gK) = i(hK)$, then $gK = hK$).

(d) Show that i is *surjective* (onto).

(e) Why do the previous parts together prove the Fundamental Homomorphism Theorem?