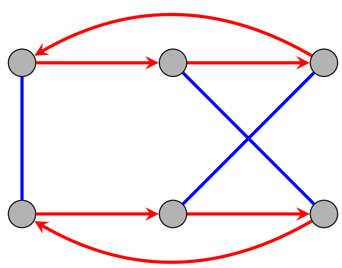
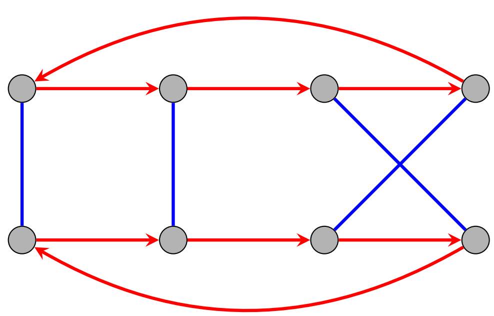
MthS 4120/6120: Abstract Algebra (Fall 2013)
Midterm 1

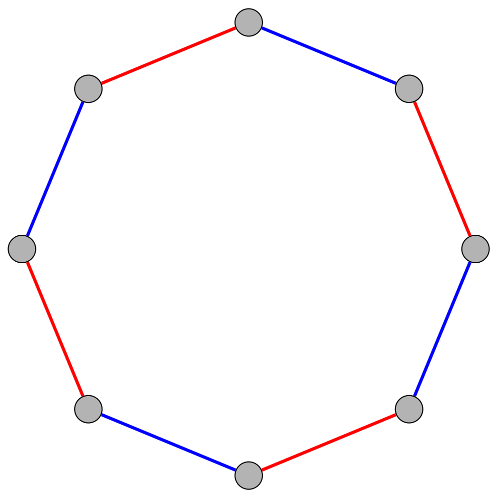
NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

(1) (6 points) Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," briefly explain why.

(a) 

(b) 

(c) 

(2) (12 points) Answer the following questions about permutations and the symmetric group.

(a) Write as a product of disjoint cycles: $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$

(b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles).

(c) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?

(d) The *order* of an element $g \in G$ is defined to be $|\langle g \rangle|$. Note that this (if finite) is also the minimum $k > 0$ such that $g^k = e$. What is the order of the element $(1\ 2\ 3)\,(4\ 5)$ in S_5?

(e) Find an element of order 20 in S_9.

(f) How many elements are there of order 18 in S_9?

(3) (8 points) Answer true or false for each statement. If false, provide a counterexample.

(a) If G is cyclic, then it is abelian.

(b) If G is abelian, then it is cyclic.

(c) Every infinite cyclic group is isomorphic to $G = (\mathbb{Z}, +)$, the integers under addition.

(d) If $|G| = \infty$, then $|\langle g \rangle| = \infty$ for all $g \in G$.

(4) (4 points) Draw a frieze pattern whose symmetry group is abelian but not cyclic. [Recall that $(\mathbb{Z}, +)$ *is* cyclic.] Draw the Cayley diagram as well. Make sure to say what generators you are using and which arrows they represent in the Cayley diagram.

(5) (8 points) The group $C_2 \times C_3$ has order 6 and consists of the set

$$C_2 \times C_3 = \{(x, y) \mid x \in \{0, 1\},\ y \in \{0, 1, 2\}\}$$

with the binary operation being componentwise addition. That is,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 \bmod 2,\ y_1 + y_2 \bmod 3)\,.$$

(a) Find the orbits of all 6 elements of $C_2 \times C_3$ and draw the orbit diagram (also called "cycle diagram").

(b) Find a minimal generating set for $C_2 \times C_3$.

(c) What familiar group is this isomorphic to? Which (and how many) of the "five families" of groups does it belong to?

(6) (8 points) When we were building up the rules of a group from scratch, we first assumed that every element had a *right inverse* (though we didn't use that exact terminology). That is, given $g \in G$, there is some h such that $gh = e$, where e is the identity. We then assumed – without proof – that $hg = e$ had to hold. In the first part of this problem you will prove this.

(a) Let G be a group and let $g \in G$. Suppose h is an element in G with the property that $gh = e$. Prove that this implies that $hg = e$. You may assume the existence of a unique *identity element* $e \in G$ sastisfying $ae = a = ea$ for all $a \in G$.

(b) Now that we know that left inverses are right inverses (and vice-versa), it makes sense to speak of "the inverse" of an element of G. That is, h is the *inverse* of g if $gh = e = hg$. Prove that every element of G has a *unique* inverse.

(7) (4 points) List *all* minimal generating sets for $D_3 = \{e,\, r,\, r^2,\, f,\, rf,\, r^2f\}$. Then do the same for $D_4 = \{e,\, r,\, r^2,\, r^3,\, f,\, rf,\, r^2f,\, r^3f\}$.