Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.1, 8.2 of *Visual Group Theory* (VGT).
- VGT Exercises 8.2, 8.5, 8.6, 8.9, 8.11, 8.13, 8.14.

Write up solutions to the following exercises.

1. Do the following steps for each mapping $\phi : G \rightarrow H$ listed below.
   
   (i) Determine whether $\phi$ is a homomorphism.
   
   (ii) Find $\ker \phi := \{g \in G \mid \phi(G) = e\}$ and $\text{im} \phi = \phi(G)$.
   
   (iii) Draw Cayley diagrams of the domain and codomain, and arrange them so one can “visually see” the cosets of $\ker \phi$ in $G$. Draw dotted lines around these cosets.
   
   (iv) Is the quotient $G/\ker \phi$ a group? If so, what is it isomorphic to?

Here is an example of Step (iii) for the map $\phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$, defined by $\phi(n) = n \mod 3$.

Now, do steps (i)–(iv) for the following maps.

(a) The map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = 4n$.
(b) The map $\phi : D_4 \rightarrow \mathbb{Z}_4$ defined by $\phi(r^k f) = k$.
(c) The map $\phi : D_4 \rightarrow \mathbb{Z}_4$ defined by $\phi(r^k f) = 2k$.
(d) The map $\phi : \mathbb{Z}_4 \rightarrow D_4$ defined by $\phi(k) = r^k$.
(e) The map $\phi : D_4 \rightarrow V_4$ defined by $\phi(r) = h$ and $\phi(f) = v$.

2. Prove that $A \times B \cong B \times A$.

3. For each part below, list all homomorphisms with the given domain and codomain.

   (a) Domain $\mathbb{Z}_{15}$ and codomain $\mathbb{Z}_4$.
   (b) Domain $\mathbb{Z}_{412}$ and codomain $\mathbb{Z}_{450}$.
   (c) Domain and codomain both $\mathbb{Z}_4$.
   (d) Domain $C_4$ and codomain $V_4$.
   (e) Domain and codomain both $V_4$. 
4. Prove that there is no embedding \( \phi: \mathbb{Z}_n \hookrightarrow \mathbb{Z} \).

5. Let \( H \leq G \), and fix \( x \in G \). Recall that we showed in class that \( xHx^{-1} \) is always a subgroup of \( G \).
   
   (a) Prove additionally that \( xHx^{-1} \cong H \). [Hint: Define a mapping from \( H \) to \( xHx^{-1} \) and prove that it is a homomorphism, one-to-one, and onto.]
   
   (b) Use Part (a) to show that \( |xy| = |yx| \) for any \( x, y \in G \).

6. Let \( \phi: G \rightarrow H \) be a homomorphism, and \( N \trianglelefteq H \).
   
   (i) Show that the set \( \phi^{-1}(N) := \{ g \in G \mid \phi(g) \in N \} \) is a subgroup of \( G \).
   
   (ii) Show that \( \phi^{-1}(N) \) is a normal subgroup of \( G \).
   
   (iii) Show by example that if \( M \trianglelefteq G \), then \( \phi(M) \) need not be a normal subgroup of \( H \).