Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.4, 8.5 of *Visual Group Theory* (VGT).
- VGT Exercises 8.15–8.18, 8.44–8.50.

Write up solutions to the following exercises.

1. For each order given below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Additionally, write each one as a product of cyclic groups organized by "elementary divisors."

   (a) 8

   (b) 54

   (c) 400

   (d) p^2q, where p and q are distinct primes.

2. The commutator subgroup of a group G is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle \,.$$

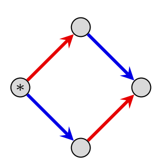
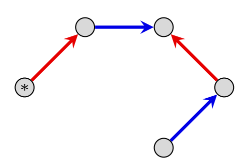
   (a) Prove that G is abelian if and only if $G' = \{e\}$.

   (b) Prove that $G' \lhd G$. [*Hint*: Take a "commutator" $c = aba^{-1}b^{-1}$ and prove that $gcg^{-1} \in G'$.]

   (c) Prove that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \lhd G} N$$

   (d) If we quotient G by G', then we are in essence, "killing" all non-abelian parts of the Cayley diagram, as shown below:

$ab = ba$  $ab \neq ba$ 

   Prove algebraically that G/G' is indeed abelian.

3. For each of the following groups G, compute its commutator subgroup G' and its abelianization G/G'. Finally, draw the subgroup lattice of G and circle every normal subgroup, and circle twice the one that is G'.

   (a) V_4

   (b) D_3

   (c) Q_4

4. Find the commutator subgroup of each of the following groups and compute its abelianization.

   (a) An abelian group A.

   (b) The alternating group A_n, for $n \geq 5$. [*Hint*: A_n is a *simple group*, which means its only normal subgroups are $\langle e \rangle$ and A_n.]

   (c) The dihedral group D_n for n even.

   (d) The dihedral group D_n for n odd.

5. For each group G, find all automorphisms and make a multiplication table of $\mathrm{Aut}(G)$. What group is it isomorphic to?

   (a) $\mathbb{Z}_7$

   (b) $\mathbb{Z}_8$

   (c) $\mathbb{Z}_{10}$

   (d) V_4

   (e) D_3