Read the following, which can all be found either in the textbook or on the course website.

- Chapters 9.3, 9.4 of *Visual Group Theory* (VGT).
- VGT Exercises 9.1, 9.4, 9.12, 9.14, 9.15, 9.21–9.27.

Write up solutions to the following exercises.

1. Let G be an unknown group of order 8. By the First Sylow Theorem, G must contain a subgroup H of order 4.

   (a) If all subgroups of G of order 4 are isomorphic to V_4, then what group must G be? Completely justify your answer.

   (b) Next, suppose that G has a subgroup $H \cong C_4$. Then G has a Cayley diagram like one of the following:

   Find all possibilities for finishing the Cayley diagram.

   (c) Label each completed Cayley diagram by isomorphism type. Justify your answer.

   (d) Make a complete list of all groups of order 8, up to isomorphism.

2. Prove that a Sylow p-subgroup of G is normal if and only if it is the unique Sylow p-subgroup of G.

3. Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$.

   (a) Show that there is no simple group of order $45 = 3^2 \cdot 5$.

   (b) Show that there is no simple group of order pq, where $p < q$ and are both prime.

   (c) Show that there is no simple group of order $12 = 2^2 \cdot 3$.

   (d) Show that there is no simple group of order $56 = 2^3 \cdot 7$.

4. Suppose that $H \lneq G$, and let $S = G/H$, the set of right cosets of H in G.

   (a) Show that if $|G|$ does not divide $[G : H]!$, then G cannot be simple. [*Hint*: Consider the action of G on S, where $\phi(g)\colon Hx \mapsto Hxg$. Prove that $\{e\} \lneq \ker\phi \lneq G$.]

   (b) Use Part (a), together with the Sylow theorems, to show that any group of order 108 cannot be simple.