Math 4120/6120: Abstract Algebra (Spring 2014)
Midterm 1

NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (12 points) Finish the following formal mathematical definitions.

   (a) A set G is a *group* if there is a binary operation $*$ on G such that the following 3 properties hold: ...

   (b) A subgroup $H \leq G$ is *normal* if ...

   (c) A group G is *cyclic* if ...

   (d) If $H \leq G$, then the left coset xH is

   $$xH = \Big\{ \qquad\qquad : \qquad\qquad \Big\}.$$

2. (6 points) Draw the subgroup lattice of the group $\mathbb{Z}_{12}$. Circle every subgroup that is normal in $\mathbb{Z}_{12}$.

3. (12 points) The *center* of a group G is the set

$$\begin{aligned} Z(G) &= \{z \in G \mid gz = zg,\ \forall g \in G\} \\ &= \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}\,. \end{aligned}$$

(a) Prove that $Z(G)$ is a subgroup of G.

(b) Prove that $Z(G)$ is a normal subgroup of G.

4. (10 points) Let G be a group and $H = \langle b, c \rangle$ a subgroup of G.

   (a) If $a \in H$, then what is $\langle a, b, c \rangle$?

   (b) If $a \notin H$ and $[G : H] = 2$, then what is $\langle a, b, c \rangle$?

   (c) If $a \notin H$, and $|G| = 48$ and $|H| = 6$, then what do you know about the order of $\langle a, b, c \rangle$? Be as specific as possible?

5. (5 points) If a is the permutation that exchanges the numbers 1 and 2, but leaves all other numbers fixed, then what is $[S_n : \langle a \rangle]$?

6. (5 points) What subgroup of $\mathbb{Z}$ is generated by 2 and 5?

7. (5 points) What is the inverse of the permutation $(1\ 2)\,(2\ 3)\,(4\ 5)$? Write your answer as a product of *disjoint* cycles.

8. (5 points) Find all left cosets of the subgroup $H = \langle (1\ 2) \rangle$ of S_3. Then find all right cosets. Is $H \lhd S_3$?

9. (16 points) Give an example of each of the following. No justification necessary.

(a) An abelian group that is not cyclic.

(b) A minimal generating set for S_5. (Use cycle notation.)

(c) A chain of subgroups $K \lhd H \lhd G$ such that $K \ntriangleleft G$. (Describe the subgroups by *specific generator(s)*, not just what group they are isomorphic to.)

(d) A group G of order 16 such that $g^2 = e$ for all $g \in G$.

(e) A group presentation for $C_2 \times C_3$.

(f) A nonabelian group G such that every proper subgroup of G is normal.

(g) A nontrivial group G such that $Z(G) = \{e\}$.

(h) A group G such that $\{e\} \lneq Z(G) \lneq G$.

10. (6 points) Prove that every element in a group G has a *unique* inverse.

11. (6 points) Prove that if $x \in H$, then $xH = H$.

12. (12 points) Let $K \leq H \leq G$ be a chain of subgroups.

    (a) Prove that if $K \lhd G$, then $K \lhd H$.

    (b) Suppose $|G| < \infty$. If $[G : H] = a$, and $[H : K] = b$, then what is $[G : K]$? Either provide a proof, OR (recommended!) provide a visual diagram that makes the answer "obvious."