Math 4120/6120: Abstract Algebra (Spring 2014)
Midterm 2

NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (10 points) Recall that a permutation $\sigma \in S_n$ is *even* if it can be written as the product of an even number of transpositions. Otherwise, it is *odd*. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication.

   Let $\phi \colon S_n \to \mathbb{R}^*$ be the homomorphism defined by: $\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$

   (a) Find $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$.

   (b) What can we conclude from the Fundamental Homomorphism Theorem? (Be as specific as possible.)

2. (10 points) Answer the following about group actions.

   (a) Carefully and completely state the Orbit-Stabilizer theorem.

   (b) Suppose a group G of order 35 acts on a set S of size 9. Prove that there must be a fixed point.

3. (20 points) Let H be a subgroup of G.

   (a) The quotient group G/H exists if and only if H ____.

   (b) The quotient group G/H consists of the set of ____.

   (c) Multiplication (i.e., the binary operation) in G/H is defined by ____________________________.

   (d) Prove that if G is abelian, then G/H is abelian.

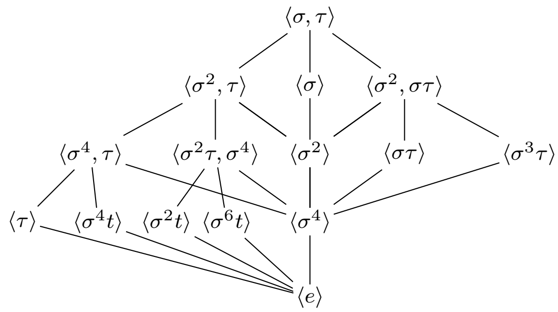
   (e) Does the converse to Part (d) hold? If yes, provide a proof. If no, provide a counterexample.

   (f) The Correspondence Theorem says that there is a one-to-one correspondence between subgroups of G/H and subgroups of ____.

4. (20 points) The *quasidihedral group* is a nonabelian group of order 16 with the following presentation:

$$QD_8 = \langle \sigma, \tau \mid \sigma^8 = 1,\ \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle \,.$$

Answer the following questions about $G = QD_8$. You should be able to do all of these from *only* the subgroup lattice of G, shown below. For full credit, *completely justify your answers!*



(a) Find the order of all subgroups of G (no justification needed). Do this by writing the order next to each subgroup in the diagram above, or otherwise making it clear from marking up the diagram.

(b) Given that the abelianiation of G is $G/G' \cong V_4$, determine which subgroup of G is the commutator subgroup, G'. [Give generator(s), not just what it is isomorphic to.]

(c) Given the knowledge that $\langle \sigma^4 \rangle \lhd G$, determine what well-known group the quotient $G/\langle \sigma^4 \rangle$ is isomorphic to.

(d) Other than G, $\langle \sigma^4 \rangle$, and $\langle e \rangle$, there are a few other subgroups that you can be absolutely certain are normal in G. Name these subgroups, and explain how you know they are normal.

(e) Determine the normalizer $N_G(\langle \sigma^4 \rangle)$.

(f) What subgroup is $\langle \sigma^4\tau, \sigma^6\tau \rangle$?

5. (20 points) Let H be a subgroup of G.

   (a) Prove that $H \cong xHx^{-1}$ for any $x \in G$. [*Hint*: Define a map and show it is a bijective homomorphism.]

   (b) Prove that if H is the unique subgroup of G of order $|H|$, then H must be normal. You may use the result of Part (a) even if you cannot prove it.

6. (20 points) Answer the following questions about group actions.

   (a) Suppose G acts on itself (that is, $S = G$) by conjugation. Then x is a fixed point if and only if

   ____________________________. (Be as specific as possible!).

   (b) Suppose G acts on itself by right multiplication. Then the stabilizer of any element $x \in G$ is

   ____________________________. (Be as specific as possible!).

   (c) Let $G = V_4 = \langle v, h \rangle$ act on its set of subgroups, $S = \Big\{\langle e \rangle, \langle v \rangle, \langle h \rangle, \langle vh \rangle, V_4\Big\}$, by conjugation. Draw the action diagram. How many orbits are there? Find the stabilizer of each $s \in S$.