Difference Equations

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Motivation: Population dynamics

Consider a population of insects that reproduces daily, of size $P(t)$:

- **birth rate** is $f \in [0, \infty)$,
- **death rate** is $d \in [0, 1]$.

This can be modeled by a simple equation:

$$\Delta P = fP - dP = (f - d)P.$$

Suppose time is **discretized**, e.g., it only takes integer values: $t = 0, 1, 2, \ldots$

Let $P_t = P(t) =$ population at time $t$.

Then $\Delta P = P_{t+1} - P_t$, from which it follows that

$$P_{t+1} = P_t + \Delta P = P_t + (f - d)P_t = (1 + f - d)P_t.$$

Letting $\lambda = 1 + f - d$ (the “finite growth rate”), we can write this as $P_{t+1} = \lambda P_t$. 
An example

Consider a population of insects that reproduces daily, with the following parameters:

- **initial population** $P_0 = 300$,
- **birth rate** $f = .03$,
- **death rate** $d = .01$.

Then the finite growth rate is $\lambda = 1 + f - d = 1.02$, and

$$P_1 = (1.02)P_0$$
$$P_2 = (1.02)P_1 = (1.02)^2 P_0$$
$$P_3 = (1.02)P_2 = (1.02)^3 P_0$$

...$

It is not difficult to see the closed-form solution $P_t = \lambda^t P_0$. This is called *exponential growth*. 
What is a difference equation?

**Definition**

Let $Q$ be a quantity defined for all $t \in \mathbb{N}$, such that $Q_{t+1} = F(Q_t)$, for some function $F$.

In the previous example: $F(x) = \lambda x$. This is called the *Malthusian model*. It is a *linear* difference equation because $F(x)$ is linear.

Let’s compare difference equations to differential equations:

- Difference equations are *discrete time, continuous space*.
- Differential equations are *continuous time, continuous space*.

**Exercise**

Can you think of a model that is discrete time and discrete space? Or continuous time and discrete space?
Which type of model to use?

Broad goals

- Find an appropriate model.
- Analyze models that naturally arise.

For example, consider the following three problems to be modeled:

1. Let $P$ be a population of $P_0 = 300$ insects with birth rate $f = .03$ and death rate $d = .01$.
2. Let $P$ be the value of an initial investment of $P_0 = 300$ dollars with fixed 2% interest rate, i.e., $\lambda = 1.02$.
3. Let $P$ be a mass of a population of bacteria that is initially $P_0 = 300$ grams, with growth rate insects with finite growth rate $\lambda = 1.02$.

Exercise

Which of these are more suited for difference equations, and which for differential equations?
Logistic equation for population growth

Realistically, a population’s growth rate isn’t constant – it depends on size. ("density dependent").

**Big idea**

Analyze $\frac{\Delta P}{P} = \text{per capita growth rate}$. 

- $P$ small: $\frac{\Delta P}{P}$ large.
- $P$ large: $\frac{\Delta P}{P}$ small.
- $P$ too large: $\frac{\Delta P}{P} < 0$.

**Assumptions:**

- Let $r$ be the growth rate when $P = 0$. [Technically, $r = \lim_{P \to 0^+} \frac{\Delta P}{P}$.] This is called the *finite intrinsic growth rate*.
- Let $M$ be the population for which $\frac{\Delta P}{P} = 0$. This is called the *carrying capacity*.
- Suppose the growth rate decreases *linearly* with $P$. 

Logistic equation for population growth

\[ \frac{\Delta P}{P} = - \frac{r}{M} P + r \]

Since the growth rate decreases \textit{linearly} with \( P \), basic algebra gives

\[ \frac{\Delta P}{P} = - \frac{r}{M} P + r = r \left(1 - \frac{P}{M}\right). \]
Logistic equation for population growth

Substituting $\Delta P = P_{t+1} - P_t$ into $\frac{\Delta P}{P} = r \left( 1 - \frac{P}{M} \right)$, followed by easy algebra yields the discrete logistic model:

$$P_{t+1} = P_t \left( 1 + r \left( 1 - \frac{P_t}{M} \right) \right).$$

Model validation

To see if this model is reasonable, the first thing to check are some simple cases:

- $P \ll M \implies 1 - \frac{P}{M} \approx 1 \implies P_{t+1} \approx (1 + r)P_t$. [Exponential growth!]
- $P \approx M \implies 1 - \frac{P}{M} \approx 0 \implies P_{t+1} \approx P_t.$

Exercise

What is $F(x)$ in the discrete logistic model? [It must satisfy $P_{t+1} = F(P_t).$]
Solutions of difference equations

Difference equations, though simple, often have *no closed form solution* for $P_t$.

However, we can plot the solutions for various initial values $P_0$.

Here are some solutions to the equation $P_{t+1} = P_t + 2P_t(1 - \frac{P_t}{10})$. 

![Graph showing population growth over time](graph.png)
Cobwebbing

Consider the difference equation $\Delta P = 0.8P_t \left(1 - \frac{P_t}{10}\right)$. Or equivalently, $P_{t+1} = F(P_t) = P_t + 0.8P_t \left(1 - \frac{P_t}{10}\right)$.

We can numerically find $P_0, P_1, P_2, \ldots$ by plotting $F(x) = x + 0.8x(1 - \frac{x}{10})$ and $y = x$ on the same axes, and then by "cobwebbing":

![Cobwebbing Diagram](image)
Consider another difference equation: \( \Delta P = 1.8P_t \left(1 - \frac{P_t}{10}\right) \). Or equivalently, 
\[ P_{t+1} = F(P_t) = P_t + 1.8P_t \left(1 - \frac{P_t}{10}\right). \]
Questions

1. Sketch a plot of several solution curves $P(t)$ for the difference equations in the previous two examples.

2. What does the spiraling behavior of this cobweb imply about the population $P(t)$?

3. How does this relate to mass-spring systems? [Hint: Think about damping.]

4. What features about a population are highlighted in the logistic equation using difference equations that do not arise using differential equations?