1. Let $X_1, X_2$ be vector spaces over a field $K$. Show that $\dim(X_1 \times X_2) = \dim X_1 + \dim X_2$.

2. Let $Y$ be a subspace of a vector space $X$. Show that $Y \times X/Y$ is isomorphic to $X$.

3. Let $K$ be a finite field. The characteristic of $K$, denoted $\text{char} K$, is the smallest positive integer $n$ for which $n1 := 1 + 1 + \cdots + 1 = 0$.

   (a) Prove that the characteristic of $K$ is prime.
   (b) Show that $K$ is a vector space over $\mathbb{Z}_p$, where $p = \text{char} K$.
   (c) Show that the order $|K|$ of $K$ (the number of elements it contains) is a prime power.
   (d) Show that if $K$ and $L$ are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then $m$ divides $n$.

4. Let $X$ be a vector space over a field $K$ and let $X'$ be the the set of linear functions from $X$ to $K$, also known as the dual space of $X$.

   (a) Let $v_1, \ldots, v_n$ be a basis for $X$. For each $i$, show there exists a unique linear map $f_i: X \to K$ such that $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for $j \neq i$.
   (b) Show that $f_1, \ldots, f_n$ is a basis for $X'$ (called the dual basis of $v_1, \ldots, v_n$).
   (c) Consider the basis $v_1 = (1, -1, 3)$, $v_2 = (0, 1, -1)$, and $v_3 = (0, 3, -2)$ of $X = \mathbb{R}^3$. Find a formula for each element of the dual basis.
   (d) Express the linear map $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, $f_1, f_2, f_3$.

5. Let $S$ be a subset of $X$. The annihilator of $S$ is the set

$$S^\perp = \{ \ell \in X' \mid \ell(s) = 0 \text{ for all } s \in S \} .$$

   (a) Show that if $S$ is a subspace of $X$, then $S^\perp$ is a subspace of $X'$.
   (b) Let $Y$ be the smallest subspace of $X$ that contains $S$. Show that $S^\perp = Y^\perp$. 