Throughout, X is assumed to be a vector space of dimension $n < \infty$.

1. Show that whenever meaningful,

   (a) $(ST)' = T'S'$

   (b) $(T+R)' = T' + R'$

   (c) $(T^{-1})' = (T')^{-1}$.

   Here, S' denotes the transpose of S. Carefully describe what you mean by "whenever meaningful" in each case.

2. Give a direct algebraic proof of $N_{T'}^{\perp} = (R_T^{\perp})^{\perp}$. (That is, don't just use the fact that $N_{T'} = R_T^{\perp}$ and take the annihilator of both sides.)

3. Let $A, B\colon X \to X$ be linear maps.

   (a) Show that if A is invertible and similar to B, then B is also invertible, and B^{-1} is similar to A^{-1}.

   (b) Show that if either A or B is invertible, then AB and BA are similar.

4. Suppose $T\colon X \to X$ is a linear map of rank 1.

   (a) Show that there exists $c \in K$ such that $T^2 = cT$.

   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

5. Suppose that $S, T\colon X \to X$ are linear maps.

   (a) Show that $\operatorname{rank}(S+T) \leq \operatorname{rank}(S) + \operatorname{rank}(T)$.

   (b) Show that $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.

   (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

   For each of these, give an explicit example showing how equality need not hold.

6. Let $T : X \to X$ be linear, with $\dim X = n$.

   (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.

   (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.

   (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.

   (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.

   (e) Show there exists a linear map $S : X \to X$ such that $ST = TS$ and $ST^{n+1} = T^n$.