Throughout, $X$ is assumed to be a vector space of dimension $n < \infty$.

1. Show that whenever meaningful,
   (a) $(ST)' = T'S'$
   (b) $(T + R)' = T' + R'$
   (c) $(T^{-1})' = (T')^{-1}.

Here, $S'$ denotes the transpose of $S$. Carefully describe what you mean by “whenever meaningful” in each case.

2. Give a direct algebraic proof of $N_{T'} = (R_{T'})^\perp$. (That is, don’t just use the fact that $N_{T'} = R_{T'}^\perp$ and take the annihilator of both sides.)

3. Let $A, B : X \to X$ be linear maps.
   (a) Show that if $A$ is invertible and similar to $B$, then $B$ is also invertible, and $B^{-1}$ is similar to $A^{-1}$.
   (b) Show that if either $A$ or $B$ is invertible, then $AB$ and $BA$ are similar.

4. Suppose $T : X \to X$ is a linear map of rank 1.
   (a) Show that there exists $c \in K$ such that $T^2 = cT$.
   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

5. Suppose that $S, T : X \to X$ are linear maps.
   (a) Show that rank$(S + T) \leq$ rank$(S) +$ rank$(T)$.
   (b) Show that rank$(ST) \leq$ rank$(S)$.
   (c) Show that dim$(N_{ST}) \leq$ dim$N_S +$ dim$N_T$.

For each of these, give an explicit example showing how equality need not hold.

6. Let $T : X \to X$ be linear, with dim$X = n$.
   (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.
   (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.
   (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.
   (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.
   (e) Show there exists a linear map $S : X \to X$ such that $ST = TS$ and $ST^{n+1} = T^n$. 