Read: Lax, Chapter 7, pages 77–89.

1. Prove that $||x|| = \max\{(x,y) : y \in K^n \text{ with } ||y|| = 1\}$.

2. Let f and g be continuous functions on the interval $[0,1]$. Prove the following inequalities.

   (a) $\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2 dt$

   (b) $\left(\int_0^1 (f(t)+g(t))^2\,dt\right)^{1/2} \leq \left(\int_0^1 f(t)^2\,dt\right)^{1/2} + \left(\int_0^1 g(t)^2 dt\right)^{1/2}$.

3. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1,2,1,1)$, $y_2 = (1,-1,0,2)$ and $y_3 = (2,0,1,1)$.

4. Let X be the vector space of all continuous real-valued functions on $[0,1]$. Define an inner product on X by
$$(f,g) = \int_0^1 f(t)g(t)\,dt\,.$$
   Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$. Find an orthonormal basis for Y.

5. Let Y be a subspace of a Euclidean space X, and $P_Y\colon X \to X$ the orthogonal projection onto Y. Prove that $P_Y^* = P_Y$.

6. Show that a matrix M is orthogonal iff its column vectors form an orthonormal set.

7. Let X be an n-dimensional real Euclidean space, and $A\colon X \to X$ a linear map. Define the map $f\colon X \to X$ by $f(x,y) = x^T Ay$. Give (with proof) necessary and suffcient conditions on A for f to be an inner product on X.