Read: Lax, Chapter 8, pages 101–120.

1. (a) Write the equation $5x_1^2 - 6x_1x_2 + 5x_2^2 = 1$ in the form $x^TAx = 1$.
   (b) Write $A = P^TDP$, where D is a diagonal matrix and P is orthogonal with determinant 1.
   (c) Sketch the graph of the equation $x^TDx = 1$ in the x_1x_2-plane.
   (d) Use a geometric argument applied to part (c) to sketch the graph of $x^TAx = 1$.
   (e) Repeat Parts (a)–(d) for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.

2. Let S be the cyclic shift mapping of $\mathbb{C}^n$, that is, $S(z_1, \ldots, z_n) = (z_n, z_1, \ldots, z_{n-1})$.
   (a) Prove that S is an isometry in the Euclidean norm.
   (b) Determine the eigenvalues and eigenvectors of S.
   (c) Verify that the eigenvectors are orthogonal.

   Hint: There are very short and elegant solutions for all three parts of this problem! You may find the last problem on HW 9 useful.

3. Let $N\colon X \to X$ be a normal mapping of a Euclidean space. Prove that $||N|| = \max |n_i|$, where the n_is are the eigenvalues of N.

4. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite. Define

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}.$$

   (a) Let $\mu = \inf\{R_{H,M}(x) \mid x \in X\}$. Show that μ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that μ and v satisfy $Hv = \mu Mv$.
   (b) Show that the constrained minimum problem

$$\min\{R_{H,M}(y) \mid (y, Mv) = 0\}$$

   has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

5. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite.
   (a) Show that there exists a basis $v_1, \ldots, v_n$ of X where each v_i satisfies an equation of the form

$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

   (b) Compute (v_i, Hv_j), and show that there is an invertible real matrix U for which $U^*MU = I$ and U^*HU is diagonal.
   (c) Characterize the numbers $\mu_1, \ldots \mu_n$ by a minimax principle.

6. Let $H, M \colon X \to X$ be self-adjoint mappings, and M positive definite.
   (a) Prove that all the eigenvalues of $M^{-1}H$ are real.
   (b) Prove that if H is positive-definite, then all the eigenvalues of $M^{-1}H$ are positive.
   (c) Show by example how Part (b) can fail if M is not positive definite.