

1. Carry out the steps outlined below for the following predator–prey model:

$$\begin{aligned}P_{t+1} &= P_t(1 + .8(1 - P_t)) - 4P_tQ_t \\ Q_{t+1} &= .9Q_t + 2P_tQ_t\end{aligned}$$

- (a) Compute the equilibria.
  - (b) Use MATLAB and the `twopop` program to make an informed guess as to whether the equilibria are stable or unstable. Print out or sketch the phase portrait.
  - (c) Linearize the model at each of the equilibria and compute eigenvalues to determine stability.
2. One approach to preventing disease spread is to simply quarantine infectives. Suppose a disease is modeled by the SIR model, but people who get the disease are health-conscience and quarantine themselves. The net result is that a fraction  $q$  of the infectives are prevented from having contacts with the susceptibles. Only  $1 - q$  of the infectives will be able to spread the disease.
    - (a) Modify the equations of the SIR model to reflect this. What value of  $q$  gives the usual SIR model?
    - (b) Quarantining can be viewed as a way of modifying the transmission coefficient. Suppose an SIR model has transmission coefficient  $\alpha$ , and a fraction  $q$  of the infectives are successfully quarantined. Then the model with quarantining is identical to a standard SIR model with some other transmission coefficient  $\alpha'$ , the *effective transmission coefficient*. Give a formula for  $\alpha'$  in terms of  $\alpha$  and  $q$ .
    - (c) Use the MATLAB program `sir` to investigate the behavior of your quarantine model for  $N = 100$ ,  $\alpha = 0.001$ , and  $\gamma = 0.05$ , and vary  $q$  from 0 to 1. Explain the qualitative behavior you see. Can you find a value of  $q$  that prevents an epidemic from occurring, regardless of  $I_0$ ? Estimate the smallest such  $q$ .
  3. Another approach to preventing disease spread is vaccination of susceptibles. Suppose a public health organization offers a vaccine for a disease modeled by the SIR model. One simple model of this situation counts each successful vaccination in the removed class throughout the duration of the model.
    - (a) Suppose that all vaccinations occur before the time  $t = 0$ . Even if this is not the case, we may assume it is – why?
    - (b) Suppose with  $N = 100$ , we have  $I_0 = 1$ , with the removed class composed of the fraction  $q$  of the population that was successfully vaccinated. Give formulas for  $S_0$  and  $R_0$  (the initial number of recovered people, *not* the basic reproductive number  $\mathcal{R}_0$ ). What value of  $q$  gives the usual SIR model?
    - (c) Repeat Part (c) from the previous problem for this situation.

4. An isolated island population of 100 individuals is exposed to a particularly deadly disease; an infected individual remains contagious until overcome by death after 4 days. We want to predict the disease's effect on the community on a daily basis. Suppose initially one individual is stricken with the disease.
- What is the removal rate  $\gamma$ ?
  - For what values of the *relative removal rate*  $\rho := \gamma/\alpha$  will an epidemic occur? Use this to determine for what values of the transmission coefficient  $\alpha$  an epidemic will occur.
  - Use a computer program such as `sir` to estimate the number of days until the epidemic peaks for the values of  $\alpha = .003, .005, .01, \text{ and } .0125$ , presenting your data in a table. How does the magnitude of  $\alpha$  relate to the time until the peak?
  - Calculate the basic reproductive numbers and the relative remove rates  $\rho$  for the values of  $\alpha$  above, adding that information to your table.
5. The following difference equation is called the *SIS model*:

$$\begin{aligned}\Delta S &= -\alpha SI + \gamma I \\ \Delta I &= \alpha SI - \gamma I.\end{aligned}$$

- What disease might be modeled well by the SIS framework?
- Use a computer program such as `sir` or `twopop` to explore the dynamics of the SIS model. Vary the parameters  $\alpha, \gamma, N, S_0$ , and  $I_0$ . Describe your findings.
- Solve for all equilibria  $(S^*, I^*)$ . Are these biologically reasonable? An equilibrium  $I^* > 0$  is called an *endemic equilibrium*. Can an SIS disease be endemic?
- Since  $S_t + I_t = N$  is constant, substitute  $I_t = N - S_t$  back into the formula for  $S_t$  and find a formula for  $S_{t+1}$  in terms of  $S_t$ . Find a formula for  $I_{t+1}$  in terms of  $I_t$ .
- For the SIR model, the threshold value  $\rho := \gamma/\alpha$ , called the *relative remove rate*, plays an important role. What does it represent? Is there an analogous threshold value for the SIS model? If so, find it. If not, explain why.
- For the SIR model, the basic reproductive number  $\mathcal{R}_0$  plays an important role. How should one define  $\mathcal{R}_0$  for the SIS model? Justify your answer.