1. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ over $\mathbb{R}$, with degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j = p(\xi_j) \quad \text{for} \quad j = 1, 2, 3\,.$$

   (a) Show that ℓ_1, ℓ_2, ℓ_3 are linearly independent functions on $\mathcal{P}_2$.

   (b) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}_2'$.

   (c) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}_2'$.

2. Let W be the subspace of $\mathbb{R}^4$ spanned by $(1, 0, -1, 2)$ and $(2, 3, 1, 1)$. Which linear functions $\ell(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W? Write your answer by giving an explicit basis of $W^\perp$.

3. Let $T\colon X \to U$ be a linear map. Prove the following:

   (a) The image of a subspace of X is a subspace of U.

   (b) The inverse image of a subspace of U is a subspace of X.

4. Prove Theorem 3.3 in Lax:

   (a) The composite of linear mappings is also a linear mapping.

   (b) Composition is distributive with respect to the addition of linear maps, that is,

$$(R + S) \circ T = R \circ T + S \circ T$$

   and

$$S \circ (T + P) = S \circ T + S \circ P\,,$$

   where $R, S\colon U \to V$ and $P, T\colon X \to U$.

5. Show that whenever meaningful,

   (a) $(ST)' = T'S'$

   (b) $(T + R)' = T' + R'$

   (c) $(T^{-1})' = (T')^{-1}$.

   Here, S' denotes the transpose of S. Carefully describe what you mean by "whenever meaningful" in each case.

6. Give a direct algebraic proof of $N_{T'}^\perp = (R_T^\perp)^\perp$. (You may use the fact that $N_{T'} = R_T^\perp$, but don't simply take the annihilator of both sides of this equation.)