

Read: Lax, Chapter 4, pages 32–43.

1. Let $T: X \rightarrow U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases \mathcal{B} for X and \mathcal{B}' for U such that the matrix of T in block form is

$$M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix, and the other blocks are either empty or contain all zeros.

2. Consider the linear map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with matrix representation $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis. What is the matrix representation of T with respect to the basis $\{(1, -1, 0), (0, 1, -1), (1, 0, 1)\}$?

3. Let \mathcal{P}_n be the vector space of all polynomials over \mathbb{R} of degree less than n .

(a) Show that the map $T: \mathcal{P}_3 \rightarrow \mathcal{P}_4$ given by

$$T(p(x)) = 6 \int_1^x p(t) dt$$

is linear. Indicate whether it is 1-1 or onto.

(b) Let $\mathcal{B}_3 = \{1, x, x^2\}$ be a basis for \mathcal{P}_3 and let $\mathcal{B}_4 = \{1, x, x^2, x^3\}$ be a basis for \mathcal{P}_4 . Find the matrix representation of T with respect to these bases.

4. Let $T: X \rightarrow X$ be linear, with $\dim X = n$.

(a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.

(b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.

(c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.

(d) Prove that $X = R_{T^n} \oplus N_{T^n}$.

(e) Show there exists a linear map $S: X \rightarrow X$ such that $ST = TS$ and $ST^{n+1} = T^n$.