

Read: Lax, Chapter 6, pages 58–69.

1. Prove the following properties of the trace function:
 - (a) $\operatorname{tr} AB = \operatorname{tr} BA$ for all $m \times n$ matrices A and $n \times m$ matrices B .
 - (b) $\operatorname{tr} AA^T = \sum a_{ij}^2$ for all $n \times n$ matrices A . The quantity is the square of the *Hilbert-Schmidt norm* of A .
2. (a) Show that if A and B are similar, then A and B have the same eigenvalues.
 (b) Is the converse of Part (a) true? Prove or disprove.
3. Let A be a 2×2 matrix over \mathbb{R} satisfying $A^T = A$. Prove that A has 2 linearly independent eigenvectors in \mathbb{R}^2 .

4. Consider the following matrices:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$

- (a) Determine the characteristic and minimal polynomials of A , B , and C .
 - (b) Determine the eigenvectors and generalized eigenvectors of A , B , and C .
5. Let A_θ be a 3×3 matrix representing a rotation of \mathbb{R}^3 through an angle θ about the y -axis.
 - (a) Find the eigenvalues for A_θ over \mathbb{C} .
 - (b) Determine necessary and sufficient conditions on θ in order for A_θ to contain three linearly independent eigenvectors in \mathbb{R}^3 . Justify your claim and interpret it geometrically.
 6. Let A be an invertible $n \times n$ matrix. Prove that A^{-1} can be written as a polynomial in degree at most $n - 1$. That is, prove that there are scalars c_i such that

$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1A + c_0I.$$

7. Let A be an $n \times n$ matrix over \mathbb{C} with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. For a vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define the *norm* of z by

$$\|z\| = \left(\sum_{i=1}^n |z_i| \right)^{1/2}.$$

- (a) Prove that if $|\lambda_i| < 1$ for all i , then $\|A^N z\| \rightarrow 0$ as $N \rightarrow \infty$ for all $z \in \mathbb{C}^n$.
- (b) Prove that if $|\lambda_i| > 1$ for all i , then $\|A^N z\| \rightarrow \infty$ as $N \rightarrow \infty$ for all nonzero $z \in \mathbb{C}^n$.