

Read: Lax, Chapter 6, pages 69–76.

1. Let A be an $n \times n$ matrix over \mathbb{C} with an eigenvalue λ of index $m \geq 2$ and corresponding eigenvector v_1 . Let v_2 be a generalized eigenvector satisfying $(A - \lambda I)v_2 = v_1$.

- (a) Prove that for any natural number N ,

$$A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1.$$

- (b) Prove that for any polynomial $q(t) \in \mathbb{C}[t]$,

$$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1,$$

where $q'(t)$ is the derivative of q .

- (c) Conjecture a formula for $q(A)v_m$, where v_1, \dots, v_m are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).

2. Consider the following matrices:

$$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

A straightforward calculation shows that the characteristic polynomials are

$$p_A(s) = p_B(s) = p_C(s) = (s - 2)^2(s - 3).$$

- (a) Determine the minimal polynomials $m_A(t)$, $m_B(t)$, and $m_C(t)$.
 - (b) Write each matrix as PJP^{-1} , where J is its Jordan canonical form.
3. Let λ be an eigenvalue of A , and let N_i be the nullspace of $(A - \lambda I)^i$. Prove that $A - \lambda I$ extends to a well-defined map $N_{i+1}/N_i \rightarrow N_i/N_{i-1}$, and that this mapping is 1-1.
 4. Let A be an $n \times n$ matrix over \mathbb{C} .
 - (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.
 - (b) Prove that if $A^k = 0$, then $A^n = 0$.
 5. Let X be an n -dimensional vector space over \mathbb{C} , and let $A, B: X \rightarrow X$ be linear maps.
 - (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ , the vector Bv is an eigenvector of A for λ .
 - (b) Show that if $\{A_1, A_2, \dots \mid A_i: X \rightarrow X\}$ is a (possibly infinite) set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i .
 - (c) Suppose that A and B are both diagonalizable. Show that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$ -matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.