Read: Lax, Appendix 15, pages 363–366.

1. Let X be an n-dimensional vector space, and $A\colon X \to X$ a linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $v_1, \dots, v_n$ be the corresponding eigenvectors of A, and let $\ell_1, \dots, \ell_n$ be the corresponding eigenvectors of the transpose $A'\colon X' \to X'$.
   (a) Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \dots, n$.
   (b) Show that if $x = a_1 v_1 + \cdots + a_n v_n$, then $a_i = (\ell_i, x)/(\ell_i, v_i)$.
   (c) Is $\ell_1, \dots, \ell_n$ necessarily the dual basis of $v_1, \dots, v_n$? Why or why not?

2. For each matrix, compute its Jordan canonical form, and give a basis of $\mathbb{C}^4$ consisting of its generalized eigenvectors.

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Let A be a matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. The *index* of $\lambda = \lambda_i$ is the minimum integer $d = d_i$ such that $N_{(A-\lambda I)^{d+1}} = N_{(A-\lambda I)^d} \supsetneq N_{(A-\lambda I)^{d-1}}$.
   (a) Prove, using the spectral theorm but *without* appealing to Jordan canonical form, that the minimal polynomial of A is
$$m_A(s) = \prod_{i=1}^{k} (s - \lambda_i)^{d_i}.$$
   (b) Give a simple proof using the Jordan canoncial form.

4. Find a list of real matrices, as long as possible, such that:
   (i) The characteristic polynomial of each matrix is $p(s) = (s-1)^5(s+1)$.
   (ii) The minimal polynomial of each matrix is $m(s) = (s-1)^2(s+1)$.
   (iii) No two matrices in the list are similar to each other.

5. Let $X \subset \mathbb{R}[x, y]$ be the space of polynomials in x, y of total degree $\leq n$. Find a basis of X, and find the Jordan canonical form of the linear map
$$A : X \longrightarrow X, \qquad f \longmapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
   [*Hint*: First find all genuine eigenvectors.]

6. This problem is about the *rational canonical form.* Consider the following matrix:

$$M_n = \begin{bmatrix} 0 & -a_0 \\ I_{n-1} & -\mathbf{a}_{n-1} \end{bmatrix} \quad \text{where } \mathbf{a}_{n-1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

   (a) Show that the characteristic polynomial of M_n is

   $$P_{M_n}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \,.$$

   Here, I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix.

   (b) Is $P_{M_n}(s)$ also the minimal polynomial? Prove or disprove.

   (c) Now, let X be a 4 dimensional vector space over $\mathbb{R}$ with basis $\{x_1, x_2, x_3, x_4\}$ and let $T : X \to X$ be a linear map such that

   $$T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.$$

   Is T diagonalizable over $\mathbb{C}$? Why or why not?