Read: Lax, Chapter 8, pages 101–120.

- 1. (a) Write the equation $5x_1^2 6x_1x_2 + 5x_2^2 = 1$ in the form $x^T A x = 1$.
 - (b) Write $A = P^T D P$, where D is a diagonal matrix and P is orthogonal with determinant 1.
 - (c) Sketch the graph of the equation $x^T Dx = 1$ in the $x_1 x_2$ -plane.
 - (d) Use a geometric argument applied to part (c) to sketch the graph of $x^T A x = 1$.
 - (e) Repeat Parts (a)–(d) for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.
- 2. Let S be the cyclic shift mapping of \mathbb{C}^n , that is, $S(z_1,\ldots,z_n)=(z_n,z_1,\ldots,z_{n-1})$.
 - (a) Prove that S is an isometry in the Euclidean norm.
 - (b) Determine the eigenvalues and eigenvectors of S.
 - (c) Verify that the eigenvectors are orthogonal.

Hint: There are very short and elegant solutions for all three parts of this problem! You may find the last problem on HW 9 useful.

- 3. Let $N: X \to X$ be a normal mapping of a Euclidean space. Prove that $||N|| = \max |n_i|$, where the n_i s are the eigenvalues of N.
- 4. Let $H, M: X \to X$ be self-adjoint mappings, and M positive definite.
 - (a) Define a scalar product on X by $\langle x,y\rangle:=(x,My)$. Prove that this is an inner product.
 - (b) Prove that all the eigenvalues of $M^{-1}H$ are real.
 - (c) Prove that if H is positive-definite, then so is $M^{-1}H$. Conclude that all eigenvalues of $M^{-1}H$ are positive.
- 5. Let $H, M: X \to X$ be self-adjoint mappings, and M positive definite. Define

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}.$$

- (a) Let $\mu = \inf\{R_{H,M}(x) \mid x \in X\}$. Show that μ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that μ and v satisfy $Hv = \mu Mv$.
- (b) Show that the constrained minimum problem

$$\min\{R_{H,M}(x) \mid (x, Mv) = 0\}$$

has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

6. Let $H, M: X \to X$ be self-adjoint mappings, and M positive definite.

(a) Show that there exists a basis v_1, \ldots, v_n of X where each v_i satisfies an equation of the form

$$Hv_i = \mu_i Mv_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- (b) Compute (v_i, Hv_j) , and show that there is an invertible real matrix U for which $U^*MU = I$ and U^*HU is diagonal.
- (c) Characterize the numbers $\mu_1, \ldots \mu_n$ by a minimax principle.