

Read: Lax, Chapter 10, pages 143–153, and Appendix 4, pages 313–316.

Let U and V be vector spaces over a field K . The *tensor product* of U and V is the set $U \otimes V$ of finite sums of the form

$$\sum c_i(u_i \otimes v_i),$$

where $c_i \in K$, $u_i \in U$, and $v_i \in V$, subject to the following relations:

$$\begin{aligned} c(u \otimes v) &= (cu) \otimes v = u \otimes (cv), \\ (u + u') \otimes v &= u \otimes v + u' \otimes v, \\ u \otimes (v + v') &= u \otimes v + u \otimes v'. \end{aligned}$$

Elements of the form $u \otimes v$ are called *pure tensors*.

1. Let U , V , and X be vector spaces over a field K . Define a map

$$\tau: U \times V \longrightarrow U \otimes V, \quad \tau(u, v) = u \otimes v.$$

- (a) Prove that τ is bilinear.
- (b) Prove that for any linear map $A: U \otimes V \rightarrow X$, the mapping $\alpha := A \circ \tau$ is bilinear from $U \times V$ to X .
- (c) Prove that for any bilinear map $\beta: U \times V \rightarrow X$, there is a unique linear mapping $A: U \otimes V \rightarrow X$ such that $\beta = A \circ \tau$. This is the *universal property* of the tensor product, and it says that any bilinear map can be “factored through” it. This is illustrated by the following commutative diagram:

$$\begin{array}{ccc} U \times V & \xrightarrow{\beta} & X \\ & \searrow \tau & \nearrow A \\ & U \otimes V & \end{array}$$

2. If $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ are bases for U and V , respectively, then it is elementary to show that the pure tensors $\{u_i \otimes v_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ span $U \otimes V$. Show that these are linearly independent, and conclude that $\dim(U \otimes V) = (\dim U)(\dim V)$. [*Hint:* Use the canonical basis $\{f_{ij}\}$ of the space of bilinear functions $U \times V \rightarrow K$, and use the universal property.]
3. Use the universal property of the tensor product to prove the following results:
 - (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);
 - (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;
 - (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.