Read: Lax, Chapter 10, pages 143–153, and Appendix 4, pages 313–316.

Let U and V be vector spaces over a field K. The tensor product of U and V is the set $U \otimes V$ of finite sums of the form

$$\sum c_i(u_i\otimes v_i)\,,$$

where $c_i \in K$, $u_i \in U$, and $v_i \in V$, subject to the following relations:

$$c(u \otimes v) = (cu) \otimes v = u \otimes (cv),$$

$$(u + u') \otimes v = u \otimes v + u' \otimes v,$$

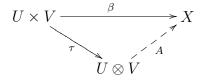
$$u \otimes (v + v') = u \otimes v + u \otimes v'.$$

Elements of the form $u \otimes v$ are called *pure tensors*.

1. Let U, V, and X be vector spaces over a field K. Define a map

$$\tau: U \times V \longrightarrow U \otimes V$$
, $\tau(u, v) = u \otimes v$.

- (a) Prove that τ is bilinear.
- (b) Prove that for any linear map $A \colon U \otimes V \to X$, the mapping $\alpha := A \circ \tau$ is bilinear from $U \times V$ to X.
- (c) Prove that for any bilinear map $\beta \colon U \times V \to X$, there is a unique linear mapping $A \colon U \otimes V \to X$ such that $\beta = A \circ \tau$. This is the *universal property* of the tensor product, and it says that any bilinear map can be "factored through" it. This is illustrated by the following commutative diagram:



- 2. If $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ are bases for U and V, respectively, then it is elementary to show that the pure tensors $\{u_i \otimes v_j \mid 1 \leq i \leq n, \ 1 \leq j \leq m\}$ span $U \otimes V$. Show that these are linearly independent, and conclude that $\dim(U \otimes V) = (\dim U)(\dim V)$. [Hint: Use the canonical basis $\{f_{ij}\}$ of the space of bilinear functions $U \times V \to K$, and use the universal property.]
- 3. Use the universal property of the tensor product to prove the following results:
 - (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);
 - (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;
 - (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.