

# Linear algebra fundamentals

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# Algebraic structures

## Definition

A **group** is a set  $G$  and associative binary operation  $*$  with:

- **closure**:  $a, b \in G$  implies  $a * b \in G$ ;
- **identity**: there exists  $e \in G$  such that  $a * e = e * a = a$  for all  $a \in G$ ;
- **inverses**: for all  $a \in G$ , there is  $b \in G$  such that  $a * b = e$ .

A group is **abelian** if  $a * b = b * a$  for all  $a, b \in G$ .

## Definition

A **field** is a set  $\mathbb{F}$  (or  $K$ ) containing  $1 \neq 0$  with two binary operations:  $+$  (addition) and  $\cdot$  (multiplication) such that:

- $\mathbb{F}$  is an abelian group under addition;
- $\mathbb{F} \setminus \{0\}$  is an abelian group under multiplication;
- The distributive law holds:  $a(b + c) = ab + ac$  for all  $a, b, c \in \mathbb{F}$ .

## Remarks

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$  (prime  $p$ ),  $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$  are all fields.
- $\mathbb{Z}$  is not a field. Nor is  $\mathbb{Z}_n$  (composite  $n$ ).
- the *additive identity* is 0, and the inverse of  $a$  is  $-a$ .
- the *multiplicative identity* is 1, and the inverse of  $a$  is  $a^{-1}$ , or  $\frac{1}{a}$ .

# Vector spaces

## Definition

A **vector space** is a set  $X$  (“vectors”) over a field  $\mathbb{F}$  (“scalars”) such that:

- (i)  $X$  is an **abelian group** under addition;
- (ii)  $+$  and  $\cdot$  are “compatible” via natural **associative** and **distributive** laws relating the two:
  - $a(bv) = (ab)v$ , for all  $a, b \in \mathbb{F}$ ,  $v \in X$ ;
  - $a(v + w) = av + aw$ , for all  $a \in \mathbb{F}$ ,  $v, w \in X$ ;
  - $(a + b)v = av + bv$ , for all  $a \in \mathbb{F}$ ,  $v, w \in X$ ;
  - $1v = v$ , for all  $v \in X$ .

## Intuition

Think of a vector space as a **set of vectors** that is:

- (i) Closed under addition and subtraction;
- (ii) Closed under scalar multiplication;
- (iii) Equipped with the “natural” associative and distributive laws.

## Proposition (exercise)

In any vector space  $X$ ,

- (i) The zero vector  $\mathbf{0}$  is unique;
- (ii)  $0x = \mathbf{0}$  for all  $x \in X$ ;
- (iii)  $(-1)x = -x$  for all  $x \in X$ .

□

# Linear maps

## Definition

A **linear map** between vector spaces  $X$  and  $Y$  over  $\mathbb{F}$  is a function  $\varphi: X \rightarrow Y$  satisfying:

- $\varphi(v + w) = \varphi(v) + \varphi(w)$ , for all  $v, w \in X$ ;
- $\varphi(av) = a\varphi(v)$ , for all  $a \in \mathbb{F}$ ,  $v \in X$ .

An **isomorphism** is a linear map that is bijective (1–1 and onto).

## Proposition

The two conditions for linearity above can be replaced by the single condition:

$$\varphi(av + bw) = a\varphi(v) + b\varphi(w), \quad \text{for all } v, w \in X \text{ and } a, b \in \mathbb{F}.$$

## Examples of vector spaces

- (i)  $K^n = \{(a_1, \dots, a_n) : a_i \in K\}$ . Addition and multiplication are defined componentwise.
- (ii) Set of functions  $\mathbb{R} \rightarrow \mathbb{R}$  (with  $K = \mathbb{R}$ ).
- (iii) Set of functions  $S \rightarrow K$  for an arbitrary set  $S$ .
- (iv) Set of polynomials of degree  $< n$ , with coefficients from  $K$ .

## Exercise

In the list of vector spaces above, (i) is isomorphic to (iv), and to (iii) if  $|S| = n$ . □

# Subspaces

## Definition

A subset  $Y$  of a vector space  $X$  is a **subspace** if it too is a vector space.

## Examples

- (i)  $Y = \{(0, a_2, \dots, a_{n-1}, 0) : a_i \in K\} \subseteq K^n$ .
- (ii)  $Y = \{\text{functions with period } T \mid \pi\} \subseteq \{\text{functions } \mathbb{R} \rightarrow \mathbb{R}\}$ .
- (iii)  $Y = \{\text{constant functions } S \rightarrow K\} \subseteq \{\text{functions } S \rightarrow K\}$ .
- (iv)  $Y = \{a_0 + a_2x^2 + a_4x^4 + \dots + a_{n-1}x^{n-1} : a_i \in K\} \subseteq \{\text{polynomials of degree } < n\}$ .

## Definition

If  $Y$  and  $Z$  are **subsets** of a vector space  $X$ , then their:

- **sum** is  $Y + Z = \{y + z \mid y \in Y, z \in Z\}$ ;
- **intersection** is  $Y \cap Z = \{x \mid x \in Y, x \in Z\}$ .

## Exercise

If  $Y$  and  $Z$  are subspaces of  $X$ , then  $Y + Z$  and  $Y \cap Z$  are also subspaces. □

# Spanning and Independence

## Definition

A **linear combination** of vectors  $x_1, \dots, x_j$  is a vector of the form  $a_1x_1 + \dots + a_jx_j$ , where each  $a_i \in K$ .

## Definition

Given a subset  $S \subseteq X$ , the subspace **spanned** by  $S$  is the set of all linear combinations of vectors in  $S$ , and denoted  $\text{Span}(S)$ .

## Exercise

For any subset  $S \subseteq X$ ,

$$\text{Span}(S) = \bigcap_{Y_\alpha \supseteq S} Y_\alpha,$$

where the intersection is taken over all subspaces of  $X$  that contain  $S$ . □

## Definition

The vectors  $x_1, \dots, x_j$  are **linearly dependent** if we can write  $a_1x_1 + \dots + a_jx_j = 0$ , where not all  $a_i = 0$ . Otherwise, the vectors are **linearly independent**.

# Spanning, linear independence, and bases

## Lemma 1.1

If  $X = \text{Span}(x_1, \dots, x_n)$ , and the vectors  $y_1, \dots, y_j \in X$  are linearly independent, then  $j \leq n$ .

## Proof outline (details to be done on the board)

Write  $y_1 = a_1x_1 + \dots + a_nx_n$ , and assume WLOG that  $a_1 \neq 0$ .

Now, “solve” for  $x_1$  and eliminate it, and conclude that

$$\text{Span}(x_1, x_2, \dots, x_n) = \text{Span}(y_1, x_2, \dots, x_n) = X$$

Repeat this process: eliminating each  $x_2, x_3, \dots$ . Note that  $j > n$  is impossible. (Why?)  $\square$

## Definition

A set  $B \subset X$  is a **basis** for  $X$  if:

- $B$  **spans**  $X$ . (is “big enough”);
- $B$  is **linearly independent**. (isn't “too big”).

## Bases

### Lemma 1.2

If  $\text{Span}(x_1, \dots, x_n) = X$ , then some subset of  $\{x_1, \dots, x_n\}$  is a basis for  $X$ .

### Proof

If  $x_1, \dots, x_n$  are linearly dependent, then we can write (WLOG; renumber if necessary)

$$x_n = a_1x_1 + \dots + a_{n-1}x_{n-1}.$$

Now,  $\text{Span}(x_1, \dots, x_{n-1}) = X$ , and we can repeat this process until the remaining set is linearly independent. □

### Definition

A vector space  $X$  is **finite dimensional** (f.d.) if it has a finite basis.

### Examples

- (i) In  $\mathbb{R}^n$ , any two vectors that don't lie on the same line (i.e., aren't scalar multiples) are linearly independent.
- (ii) In  $\mathbb{R}^3$ , any three vectors are linearly independent iff they do not lie on the same plane.
- (iii) Any two vectors in  $\mathbb{R}^2$  that aren't scalar multiples form a basis.



# Dimension

## Theorem / Definition 1.3

All bases for a f.d. vector space have the same cardinality, called the **dimension** of  $X$ .

### Proof

Let  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  be two bases for  $X$ . By Lemma 1.1,  $m \leq n$  and  $n \leq m$ .  $\square$

## Theorem 1.4

Every linear independent set of vectors  $y_1, \dots, y_j$  in a finite-dimensional vector space  $X$  can be **extended** to a basis of  $X$ .

### Proof

If  $\text{Span}(y_1, \dots, y_j) \neq X$ , then find  $y_{j+1} \in X$  not in  $\text{Span}(y_1, \dots, y_j)$ , add it to the set and repeat the process.

This will terminate in less than  $n = \dim X$  steps because otherwise,  $X$  would contain more than  $n$  linearly independent vectors.  $\square$

## Complements and direct sums

### Theorem 1.5

- (a) Every subspace  $Y$  of a finite-dimensional vector space  $X$  is finite-dimensional.
- (b) Every subspace  $Y$  has a **complement** in  $X$ : another subspace  $Z$  such that every vector  $x \in X$  can be written uniquely as

$$x = y + z, \quad y \in Y, z \in Z, \quad \dim X = \dim Y + \dim Z.$$

### Proof

Pick  $y_1 \in Y$  and extend this to a basis  $y_1, \dots, y_j$  of  $Y$ . By Lemma 1.1,  $j \leq \dim X < \infty$ .

Extend this to a basis  $y_1, \dots, y_j, z_{j+1}, \dots, z_n$  of  $X$  [and define  $Z := \text{Span}(z_{j+1}, \dots, z_n)$ ].

Clearly,  $Y$  and  $Z$  are complements, and  $\dim X = n = j + (n - j) = \dim Y + \dim Z$ .  $\square$

### Definition

$X$  is the **direct sum** of subspaces  $Y$  and  $Z$  that are complements of each other.

More generally,  $X$  is the direct sum of subspaces  $Y_1, \dots, Y_m$  if every  $x \in X$  can be expressed uniquely as

$$x = y_1 + \dots + y_m, \quad y_i \in Y_i.$$

We denote this as  $X = Y_1 \oplus \dots \oplus Y_m$ .

## Direct products

### Definition

The **direct product** of  $X_1$  and  $X_2$  is the vector space

$$X_1 \times X_2 := \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\},$$

with addition and multiplication defined componentwise.

### Proposition

- $\dim(Y_1 \oplus \cdots \oplus Y_m) = \sum_{i=1}^m \dim Y_i$ ;
- $\dim(X_1 \times \cdots \times X_m) = \sum_{i=1}^m \dim X_i$ .

### Example

Let  $X = \mathbb{R}^4$ ,  $Y_1 = \{(a, b, 0, 0) : a, b \in \mathbb{R}\}$ ,  $Y_2 = \{(0, 0, c, d) : c, d \in \mathbb{R}\}$ ,  $X_1 = X_2 = \mathbb{R}^2$ .

Clearly,  $X = Y_1 \oplus Y_2$ , since  $(a, b, c, d) = (a, b, 0, 0) + (0, 0, c, d)$  [uniquely].

$$X_1 \times X_2 = \left\{ ((a, b), (c, d)) : (a, b) \in \mathbb{R}^2, (c, d) \in \mathbb{R}^2 \right\} \cong \{(a, b, c, d) : a, b, c, d \in \mathbb{R}\} = X.$$

## Direct sums vs. direct products

In the finite-dimensional cases, there is no difference (up to isomorphism) of direct sums vs. direct products.

Not the case when  $\dim X = \infty$ . Consider the vector space:

$$X = \mathbb{R}^\infty := \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$$

and the following subspaces:

$$X_1 = \{(a_1, 0, 0, 0, \dots) : a_1 \in \mathbb{R}\}, \quad X_2 = \{(0, a_2, 0, 0, \dots) : a_2 \in \mathbb{R}\}, \quad \text{and so on.}$$

Elements in the subspace  $X_1 \oplus X_2 \oplus X_3 \oplus \dots$  of  $X$  are **finite sums**

$$x = x_{i_1} + x_{i_2} + \dots + x_{i_k}, \quad x_{i_j} \in X_{i_j}.$$

Thus, we can write the direct sum as follows:

$$X_1 \oplus X_2 \oplus X_3 \oplus \dots = \{(a_1, \dots, a_k, 0, 0, \dots) : a_i \in \mathbb{R}, k \in \mathbb{Z}\} \subsetneq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$$

- Elements in the **direct product** are **sequences**, e.g.,  $x = (1, 1, 1, \dots)$ .
- Elements in the **direct sum** are **finite sums**, e.g.,  $x = 3e_1 - 5.25e_4 + 78e_{11}$ .

## Congruence of subspaces

Sums and products “multiply” vector spaces. We can also “divide” by a subspace.

### Definition

If  $Y$  is a subspace of  $X$ , then two vectors  $x_1, x_2 \in X$  are **congruent modulo  $Y$** , denoted  $x_1 \equiv x_2 \pmod{Y}$ , if  $x_1 - x_2 \in Y$ .

### Proposition (exercise)

Congruence modulo  $Y$  is an **equivalence relation**, i.e., it is:

- (i) **symmetric**:  $x \equiv y$  implies  $y \equiv x$ ;
- (ii) **reflexive**:  $x \equiv x$  for all  $x \in X$ ;
- (iii) **transitive**:  $x \equiv y$  and  $y \equiv z$  implies  $x \equiv z$ . □

The equivalence classes are called **congruence classes mod  $Y$** , or **cosets**. Denote the class containing  $x$  by  $\{x\}$ . [Sometimes written  $\bar{x}$  or  $x + W := \{x + w : w \in W\}$ .]

### Example

Let  $X = \mathbb{R}^3$ ,  $Y = \{(x, y, 0) : x, y \in \mathbb{R}\} = xy\text{-plane}$ ,  $Z = \{(0, 0, z) : z \in \mathbb{R}\} = z\text{-axis}$ .

- $v \equiv w \pmod{Y}$  if they lie on the same horizontal plane.
- $v \equiv w \pmod{Z}$  if they lie on the same vertical line.

## Quotient spaces

Let  $X/Y$  denote the set of equivalence classes in  $X$ , modulo  $Y$ .

This can be made into a vector space by defining addition and scalar multiplication as follows:

$$\{x\} + \{z\} := \{x + z\}, \quad a\{x\} := \{ax\}.$$

Need to check that this is **well-defined**, i.e., that it is *independent of the choice of representative* from the classes.

This means showing (HW exercise) that if  $x_1 \equiv x_2 \pmod{Y}$  and  $z_1 \equiv z_2 \pmod{Y}$ , then

$$\{x_1\} + \{z_1\} = \{x_2\} + \{z_2\}, \quad a\{x_1\} = a\{x_2\}.$$

### Definition

The vector space  $X/Y$  is called the **quotient space** of  $X$  modulo  $Y$ .

### Alternate notations

Since  $\{x\}$  is sometimes written  $\bar{x}$ , or  $x + W := \{x + w : w \in W\}$ , then addition and multiplication becomes:

- $\bar{x} + \bar{z} = \overline{x + z}$ , and  $a\bar{x} = \overline{ax}$ ;
- $(x + W) + (z + W) = x + z + W$ , and  $a(x + W) = ax + W$ .

## Dimension of quotient spaces

### Theorem 1.6

If  $Y$  is a subspace of a finite-dimensional vector space  $X$ , then  $\dim Y + \dim X/Y = \dim X$ .

### Proof

Let  $y_1, \dots, y_j$  be a basis for  $Y$ . Extend this to a basis  $y_1, \dots, y_j, x_{j+1}, \dots, x_n$  of  $X$ .

**Claim:**  $\{x_{j+1}\}, \dots, \{x_n\}$  is a basis of  $X/Y$ .

- Show this spans  $X/Y$ :

Pick  $\{x\}$  in  $X/Y$  and write  $x = \sum_{i=1}^j a_i y_i + \sum_{k=j+1}^n b_k x_k$ . By definition,

$$\{x\} = \left\{ \sum a_i y_i + \sum b_k x_k \right\} = \sum a_i \{y_i\} + \sum b_k \{x_k\} = \sum b_k \{x_k\}.$$

- Show this is linearly independent:

Suppose  $\sum_{k=j+1}^n c_k \{x_k\} = \{0\}$ , which means  $\sum c_k x_k = y$  for some  $y \in Y$ .

Write  $y = \sum_{i=1}^j d_i y_i$ , and so  $\sum c_k x_k - \sum d_i y_i = 0$ , and hence all  $c_k, d_i = 0$  (Why?).  $\square$

### Corollary

If a subspace  $Y$  of a finite-dimensional space  $X$  has  $\dim Y = \dim X$ , then  $Y = X$ .  $\square$

## Dimension of sums

### Theorem 1.7

Let  $U, V$  be subspaces of a finite-dimensional space  $X$  with  $U + V = X$ . Then

$$\dim X = \dim U + \dim V - \dim(U \cap V).$$

### Proof

Let  $W = U \cap V$ . The result trivially holds when  $W = \{0\}$  (Theorem 1.5).

Define  $\bar{U} = U/W$ ,  $\bar{V} = V/W$  and  $\bar{X} = X/W$ .

Note that  $\bar{U} \cap \bar{V} = \{0\}$  (why?), and  $\bar{X} = \bar{U} + \bar{V}$ , and so  $\dim \bar{X} = \dim \bar{U} + \dim \bar{V}$  (Theorem 1.5).

By Theorem 1.6:

$$\begin{aligned}\dim \bar{X} &= \dim X - \dim W \\ \dim \bar{U} &= \dim U - \dim W \\ \dim \bar{V} &= \dim V - \dim W\end{aligned}$$

Therefore,  $(\dim X - \dim W) = (\dim U - \dim W) + (\dim V - \dim W)$ .

From which it easily follows that  $\dim X = \dim U + \dim V - \dim W$ . □



## An interesting example

Let  $X$  be the set of all (twice-differential) functions  $x(t)$  that satisfy the second order differential equation  $\frac{d^2}{dt^2}x + x = 0$ .

If  $x_1(t)$ ,  $x_2(t)$  are solutions, then so are  $x_1(t) + x_2(t)$  and  $cx_1(t)$ . Thus  $X$  is a vector space.

Solutions describe the motion of a mass-spring system (**simple harmonic motion**). A particular solution is determined completely by specifying:

$$x(0) = x_0 \quad (\text{initial position}) \quad x'(0) = v_0 \quad (\text{initial velocity}).$$

Thus, we can describe an element  $x(t) \in X$  by a pair  $(x_0, v_0)$ , where  $x_0, v_0 \in \mathbb{R}$  (or in  $\mathbb{C}$ ).

This defines an **isomorphism**  $X \rightarrow \mathbb{C}^2$ , by  $x(t) \mapsto (x(0), x'(0))$ .

Note that  $\cos x$  and  $\sin x$  are two **linearly independent** solutions, so the **general solution** to this ODE is  $a \cos x + b \sin x$ ;  $a, b \in \mathbb{C}$ .

Said differently,  $\{\cos x, \sin x\}$  is a **basis for the solution space of  $x'' + x = 0$** .

Note that  $\cos x + i \sin x = e^{ix}$  and  $\cos x - i \sin x = e^{-ix}$  are linearly independent, and so  $\{e^{ix}, e^{-ix}\}$  is another basis! Thus, the general solution can be written as  $C_1 e^{ix} + C_2 e^{-ix}$  instead!