Read: Chapter 1: Mechanisms of gene regulation: Boolean network models of the lactose operon in *Escherichia coli*, by R. Robeva, B. Kirkwood, and R. Davis, pages 1–35.

Do: Create an account on the Sage Math Cloud (`https://cloud.sagemath.org`).

1. Consider the following system of polynomial equations:

$$\begin{aligned} x^2 + y^2 + xyz &= 1 \\ x^2 + y + z^2 &= 0 \\ x - z &= 0 \end{aligned}$$

   To compute a Gröbner basis for this system over $\mathbb{Q}$, type the following commands into Sage, one-by-one, and press Shift+Enter after each one:

```
P.<x,y,z> = PolynomialRing(QQ, 3, order='lex'); P
I = ideal(x^2+y^2+x*y*z-1, x^2+y+z^2, x-z); I
B = I.groebner_basis(); B
```

   (a) For the system above, use the Gröbner basis you just computed to write a simpler system of polynomial equations that has the same set of solutions. Solve that system *by hand* (it's not hard) to find all *real-valued* solutions to the original system.
   (b) Next, solve the original system but over the binary field, $\mathbb{F}_2 = \{0, 1\}$. For this, you need to replace `RR` with `GF(2)` in the Sage code.
   (c) Now, solve the original system but over the ternary field, $\mathbb{F}_3 = \{0, 1, 2\}$.

2. Repeat the previous problem for this system of polynomial equations:

$$\begin{aligned} x^2y - z^3 &= 0 \\ 2xy - 4z &= 1 \\ z - y^2 &= 0 \\ x^3 - 4yz &= 0 \end{aligned}$$

3. Consider the following simple model of the *lac* operon:

$$\begin{aligned} f_M &= \overline{R} & \qquad f_R &= \overline{A} \\ f_P &= M & \qquad f_A &= L \wedge B \\ f_B &= M & \qquad f_L &= P \end{aligned}$$

   For this problem, make the convention that $(x_1, x_2, x_3, x_4, x_5, x_6) = (M, P, B, R, A, L)$.

   (a) Justify each function in a single sentence. What other assumptions are made in this model? (E.g., presence or absense of external lactose and glucose?)

(b) Write each function as a polynomial over $\mathbb{F}_2 = \{0, 1\}$. Then, write out the system of equations $\{f_i + x_i = 0,\ i = 1, \dots, 6\}$, whose solutions are the fixed points of the Boolean network.

(c) Go into Sage and type the following command:

```
P.<x1,x2,x3,x4,x5,x6> = PolynomialRing(GF(2), 6, order='lex'); P
```

Now, define an ideal I generated by the six polynomials, $f_i + x_i$, from Part (b). Use Sage to compute the Gröbner basis of this ideal, and include a print-out of a screenshot.

(d) The Gröbner basis describes a simpler system of equations with the same solutions as the original. Write out this system and then solve it by hand to determine the fixed points of the Boolean network.

(e) Compute the entire phase space of your model with the help of either the *Analysis of Dynamic Algebraic Models* (ADAM) toolbox, at `http://adam.plantsimlab.org/`, or *TURING: Algorithms for computation with finite dynamical systems*, at `http://www.discretedynamics.org`. Include a print-out of a screenshot. Are there any periodic points that are not fixed points?