1. Let $T\colon X \to U$ be a linear map. Prove the following:
   (a) The image of a subspace of X is a subspace of U.
   (b) The inverse image of a subspace of U is a subspace of X.

2. Let X and U be vector spaces, and suppose that Y is a subspace of X. Let $Q\colon X \to X/Y$ be the canonical quotient map sending $x \overset{Q}{\longmapsto} \{x\}$, and let $T\colon X \to U$ be a linear map. Give necessary and sufficient conditions for the existence of a unique linear map $S\colon X/Y \to U$ such that $T = S \circ Q$. When this happens, the map T is said to *factor through* the quotient space, as shown by the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q"'] & & U \\ & X/Y \arrow[ur, dashed, "S"'] & \end{tikzcd}$$

   Prove all of your claims.

3. Suppose $T\colon X \to X$ is a linear map of rank 1.
   (a) Show that there exists $c \in K$ such that $T^2 = cT$.
   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

4. Suppose that $S, T\colon X \to X$ are linear maps.
   (a) Show that $\operatorname{rank}(S+T) \leq \operatorname{rank}(S) + \operatorname{rank}(T)$.
   (b) Show that $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.
   (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

   For each of these, give an explicit example showing how equality need not hold.

5. Show that whenever meaningful,
   (a) $(ST)' = T'S'$
   (b) $(T+R)' = T' + R'$
   (c) $(T^{-1})' = (T')^{-1}$.

   Here, S' denotes the transpose of S. Carefully describe what you mean by "whenever meaningful" in each case.

6. Give a direct algebraic proof of $N_{T'}^{\perp} = (R_T^{\perp})^{\perp}$. (You may use the fact that $N_{T'} = R_T^{\perp}$, but don't simply take the annihilator of both sides of this equation.)