Read: Lax, Chapter 6, pages 58–69.

1. Prove the following properties of the trace function:
   (a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $m \times n$ matrices A and $n \times m$ matrices B.
   (b) $\mathrm{tr}(AA^T) = \sum a_{ij}^2$ for all $n \times n$ matrices A. This quantity is the square of the *Hilbert-Schmidt norm* of A.

2. Let A be an $n \times n$ matrix over $\mathbb{C}$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. For a vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define the *norm* of z by
$$||z|| = \Big(\sum_{i=1}^{n} |z_i|^2\Big)^{1/2}.$$
   (a) State and prove necessary and sufficient conditions for $\lim\limits_{N\to\infty} ||A^N z|| = 0$ for all $z \in \mathbb{C}$.
   (b) Prove that if $|\lambda_i| > 1$ for each i, then $\lim\limits_{N\to\infty} ||A^N z|| = \infty$ for all $z \neq 0$. Soon, after developing more analytical techniques, we will formulate and prove a necessary and sufficient condition for this to hold.

3. Suppose that $B = PAP^{-1}$, and A has eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $v_1, \ldots, v_n$. What are the eigenvalues and eigenvectors of B? Prove your claims.

4. Let X be an n-dimensional vector space, and $A \colon X \to X$ a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors of A, and let $\ell_1, \ldots, \ell_n$ be the corresponding eigenvectors of the transpose $A' \colon X' \to X'$.
   (a) Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \ldots, n$.
   (b) Explain why every $x \in X$ can be written as $x = a_1 v_1 + \cdots + a_n v_n$, and derive a formula for a_i.
   (c) Is $\ell_1, \ldots, \ell_n$ necessarily the dual basis of $v_1, \ldots, v_n$? Why or why not?

5. Consider the following matrices:
$$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$
   A straightforward calculation shows that the characteristic polynomials are
$$p_A(s) = p_B(s) = p_C(s) = (s-2)^2(s-3).$$
   (a) Find the eigenvectors and the minimal polynomials of each matrix
   (b) Find a basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ where $Bv_1 = 3v_1$, $Bv_2 = 2v_2$, and $(B - 2I)v_3 = v_2$. Write the matrix of this linear map with respect to this new basis.
   (c) Repeat the previous step for the matrix C.