Read: Lax, Chapter 6, pages 69–76.

1. Let A be an invertible $n \times n$ matrix. Prove that A^{-1} can be written as a polynomial in degree at most $n-1$. That is, prove that there are scalars c_i such that

$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1 A + c_0 I.$$

2. Let λ be an eigenvalue of A, and let N_i be the nullspace of $(A-\lambda I)^i$. Elements of N_i are called *generalized eigenvectors* of λ. The special case of $i=1$ are the ordinary ("genuine") eigenvectors. Prove that $A-\lambda I$ extends to a well-defined map $N_{i+1}/N_i \longrightarrow N_i/N_{i-1}$, and that this mapping is 1–1.

3. Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ and corrresponding eigenvector v_1. Let v_2 be a generalized eigenvector satisfying $(A-\lambda I)v_2 = v_1$.

   (a) Prove that for any natural number N,

$$A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1\,.$$

   (b) Prove that for any polynomial $q(t) \in \mathbb{C}[t]$,

$$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1\,,$$

   where $q'(t)$ is the derivative of q.

   (c) Conjecture a formula for $q(A)v_m$, where $v_1, \ldots, v_m$ are generalized eigenvectors of A with $(A-\lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).

4. Do the following for the matrix A below, and then repeat it for B:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Find the characteristic polynomial and all (genuine) eigenvectors.

   (b) For each eigenvalue λ, compute $\dim N_{(A-\lambda I)^j}$ for $j = 1, 2, 3, \ldots$.

   (c) Find a basis of $\mathbb{C}^4$ consisting of generalized eigenvectors.

   (d) Factor the matrix as a product PJP^{-1}, where the columns of P are the generalized eigenvectors and J is the matrix of the linear map with respect to this new basis.