*Read*: Lax, Appendix 15, pages 363–366.

1. Let $X \subset \mathbb{R}[x]$ be the space of polynomials of degree $< n$ and consider the linear map
$$D : X \longrightarrow X, \qquad f \longmapsto \frac{df}{dx}.$$
Find the eigenvalues of $D$, and then find a basis $f_0, \dots, f_{n-1}$ of $X$ consisting of generalized eigenvectors of $D$ so that the matrix $J$ with respect to this basis is in Jordan canonical form. Write down $J$.

2. Let $A$ be a $7 \times 7$ matrix over $\mathbb{C}$ with minimal polynomial $m(s) = (s-1)^3(s-2)^2$.
   (a) List all possible Jordan canonical forms of $A$ of to similarity.
   (b) For each matrix from Part (a), find the rank of $(A-I)^k$ and $(A-2I)^k$, for $k \in \mathbb{N}$.

3. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. The matrix $A$ is *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$.
   (a) Prove that if $A$ is nilpotent, then $A^n = 0$.
   (b) Prove that if $A$ is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine $A$ up to similarity.
   (c) Suppose $A$ and $B$ are $6 \times 6$ nilpotent matrices with the same minimal polynomial and $\dim N_A = \dim N_B$. Prove that $A$ and $B$ are similar. Show by example that this can fail for $7 \times 7$ matrices.

4. Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$. The matrix $A$ is *idempotent* if $A^2 = A$.
   (a) Prove that if $A^k = A$ for some integer $k > 1$, then $A$ is diagonalizable.
   (b) Prove that idempotent matrices are similar if and only if they have the same trace.
   (c) Prove that if $A$ and $B$ are idempotent and $B = UAV$ holds for some invertible maps $U, V \colon X \to X$, then $A$ and $B$ are similar.

5. Let $X$ be an $n$-dimensional vector space over $\mathbb{C}$, and let $A, B \colon X \to X$ be linear maps.
   (a) Prove that if $AB = BA$, then for any eigenvector $v$ of $A$ with eigenvalue $\lambda$, the vector $Bv$ is an eigenvector of $A$ for $\lambda$.
   (b) Show that if $\{A_1, \dots, A_k \mid A_i \colon X \to X\}$ is a set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every $A_i$.
   (c) Suppose that $A$ and $B$ are both diagonalizable. Prove that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$-matrix $P$ such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.