Read: Lax, Chapter 7, pages 89–100.

1. Let X be the space of continuous complex-valued functions on $[-1,1]$ and define an inner product on X by
$$(f,g) = \int_{-1}^{1} f(s)\overline{g(s)}\,ds\,.$$
Let $m(s)$ be a continuous function of absolute value 1, that is, $|m(s)| = 1$, $-1 \leq s \leq 1$. Define M to be multiplication by m:
$$(Mf)(s) = m(s)f(s)\,.$$
Show that M is unitary.

2. Let A be a linear map of a finite-dimensional complex inner product space X.
   (a) A matrix is *normal* if $AA^* = A^*A$. It is unitarily similar to a diagonal matrix if $A = U^*DU$ for a diagonal matrix D and unitary matrix U. Show that these conditions are equivalent.
   (b) Prove that if A is normal then it has a square-root, that is, a matrix B such that $A = B^2$. Is B necessarily normal? Unique?
   (c) Suppose that A is diagonalizable. Prove that A is normal if and only if each eigenvector of A is an eigenvector of A^*.

3. Let S be the cyclic shift mapping of $\mathbb{C}^n$, that is, $S(z_1,\dots,z_n) = (z_n, z_1, \dots, z_{n-1})$.
   (a) Prove that S is unitary.
   (b) Determine the eigenvalues and eigenvectors of S.
   (c) Find an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of S.

4. (a) Write the equation $5x_1^2 - 6x_1x_2 + 5x_2^2 = 1$ as $x^TAx = 1$.
   (b) Write $A = P^TDP$, where D is a diagonal matrix and P is orthogonal with determinant 1.
   (c) Sketch the graph of the equation $x^TDx = 1$ in the x_1x_2-plane.
   (d) Use a geometric argument applied to part (c) to sketch the graph of $x^TAx = 1$.

5. Repeat the previous problem for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.