Read: Lax, Chapter 8, pages 101–120.

1. Let $N\colon X \to X$ be a normal mapping of an inner product space. Prove that $||N|| = \max|n_i|$, where the n_is are the eigenvalues of N.

2. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite. Define an inner product on X by $\langle x, y\rangle := (x, My)$.
   (a) Prove that all eigenvalues of $M^{-1}H$ are real.
   (b) Prove that if H is positive-definite, then so is $M^{-1}H$. Conclude that all eigenvalues of $M^{-1}H$ are positive.

3. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite. Define
$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}\,.$$
   (a) Let $\mu = \inf\{R_{H,M}(x) \mid x \in X\}$. Show that μ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that μ and v satisfy $Hv = \mu Mv$.
   (b) Show that the constrained minimum problem
$$\min\big\{R_{H,M}(x) \mid (x, Mv) = 0\big\}$$
   has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

4. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive definite.
   (a) Show that there exists a basis $v_1, \dots, v_n$ of X where each v_i satisfies an equation of the form
$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
   (b) Compute (v_i, Hv_j), and show that there is an invertible matrix U for which $U^*MU = I$ and U^*HU is diagonal.
   (c) Characterize the numbers $\mu_1, \dots \mu_n$ by a minimax principle.