3. Second order differential equations

We will consider equations of the form $y'' = f(t, y, y')$.

A solution is any function $y(t)$ s.t. $y''(t) = f(t, y(t), y'(t))$.

Motivating example: $F = ma$ (Newton's 2^{nd} law of motion).

Force (could be gravitational, mechanical, etc) can be a function of time, displacement $x(t)$, and velocity $x'(t)$.

$F(t, x, x') = m\,x''(t)$.

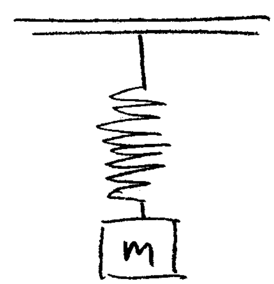
Ex 1: Gravity ("constant" force): $\boxed{m x''(t) = -mg}$

Ex 2: Spring  (at rest)

Hooke's law: Restoring force $R(x) = -kx \Rightarrow \boxed{m x''(t) = -kx}$

Think: "Force is proportional to how much we stretch or compress."

Ex 3: Now, suppose the weight is hanging.



Forces add, so $F = R(x) + (\text{Grav. force})$

$\boxed{m x'' = -kx + mg}$ (Note: Why $+mg$?)

Ex 4: Suppose there's also a damping force (springs never "bounce forever").

This is like air resistance:
- Proportional to velocity
- Acts against the direction of motion.

Thus, $D(x') = -\mu x'$, μ const.

Forces add, so $F = D(x') + R(x) + mg \Rightarrow \boxed{mx'' = -\mu x' - kx + mg}$

There are 2 "general techniques" for analyzing 2^{nd} order ODE's:

(i) Solving them directly

(ii) Turning them into systems of 1^{st} order ODE's.

Example: $y'' + 3ty' + 2y = \sin t$, Let $v = y'$, so $v' = y''$

We now have $\begin{cases} v' + 3tv + 2y = \sin t \\ v = y' \end{cases}$

We'll do (i) first, because it's an extension of what we've done for 1^{st} order ODE's. (Section 4 will be devoted to (ii); linear systems).

A linear 2^{nd} order ODE has the form: $y'' + p(t)y' + q(t)y = g(t)$

A homogeneous (linear) 2^{nd} order ODE: $y'' + p(t)y' + q(t)y = 0$

* Big idea The general solution to a linear 2^{nd} order ODE is

$$\boxed{y(t) = \underbrace{C_1 y_1(t) + C_2 y_2(t)}_{y_h(t)} + y_p(t)}$$ (a 2-parameter family)

where $y_p(t)$ is any particular solution.

Remark: $y_p(t) = 0$ is a solution of the ODE if and only if it is homogeneous. (why?)

Examples:

- Find the general sol'n to $\boxed{y'' = k^2 y}$

  Observe that $y_1(t) = e^{kt}$ works, as does $y_2(t) = e^{-kt}$.

  Thus, the general sol'n is $\boxed{y(t) = C_1 e^{kt} + C_2 e^{-kt}}$

- Find the general sol'n to $\boxed{y'' = -k^2 y}$

  Observe that $y_1(t) = \cos kt$ works, as does $y_2(t) = \sin kt$.

  Thus, the general sol'n is $\boxed{y(t) = A\cos kt + B \sin kt}$

- Find the general sol'n to $\boxed{y'' - 3y' + 2y = 0}$

  What might be a good guess?

  Try $y(t) = e^{rt}$ where r is some constant

  Solve for r:
  $$y = e^{rt}$$
  $$y' = re^{rt}$$
  $$y'' = r^2 e^{rt}$$

  Plug back into $y'' - 3y' + 2y = 0$:
  $$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0$$
  $$e^{rt}(r^2 - 3r - 2) = 0$$
  $$e^{rt}(r-1)(r-2) = 0 \implies r = 1 \text{ or } 2$$

  Thus, we've found two sol'ns: $y_1(t) = e^t$, $y_2(t) = e^{2t}$,

  so the general sol'n is $\boxed{y(t) = C_1 e^t + C_2 e^{2t}}$

Question: What if we have a repeated root?

e.g., $\boxed{y'' - 6y' + 9y = 0}$

Again, guess $\left.\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}\right\}$ $\begin{aligned} &r^2 e^{rt} - 6re^{rt} + 9e^{rt} = 0 \\ &e^{rt}(r^2 - 6r + 9) = 0 \\ &(r-3)^2 = 0 \Rightarrow r = 3. \end{aligned}$

We've determined that $y_1(t) = C_1 e^{3t}$ is a solution.

But we need one more!

Try $y(t) = v(t)\, e^{3t}$, and solve for $v(t)$.

If $y = ve^{3t}$, then $y' = 3e^{3t}v + e^{3t}v'$, and

$$y'' = 3(3e^{3t}v + e^{3t}v') + (3e^{3t}v' + e^{3t}v'')$$
$$= 9e^{3t}v + 6e^{3t}v' + e^{3t}v''$$

Plug back into ODE:

$$\underbrace{(9e^{3t}v + 6e^{3t}v' + e^{3t}v'')}_{y''} - 6\underbrace{(3e^{3t}v + e^{3t}v')}_{y'} + 9\underbrace{(e^{3t}v)}_{y} = 0.$$

$$v''e^{3t} = 0 \Rightarrow v'' = 0 \Rightarrow v(t) = Ct + D$$

Conclusion: e^{3t} is a sol'n, and $(Ct + D)$ is a sol'n for <u>any</u> C & D, so let's choose $C = 1$, $D = 0$, so $v(t) = t$.

Now, $y_1(t) = e^{3t}$, $y_2(t) = v(t)\, e^{3t} = te^{3t}$, so the general solution is $y(t) = C_1 e^{3t} + C_2 t e^{3t}$

Question: What if we have complex roots?

i.e, suppose $y'' + py' + qy = 0$, and the roots to the "characteristic equation" are $r_{1,2} = a \pm bi$

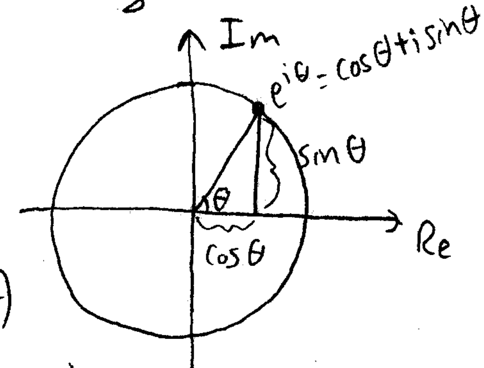
We have 2 solutions: $y_1(t) = e^{(a+bi)t}$, $y_2(t) = e^{(a-bi)t}$

Thus, the general solution is $y(t) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$.

What do these functions look like???

There's indeed a "better way" to write this general sol'n

* Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$



Recall: $\cos(-x) = x$, $\sin(-x) = -\sin x$

$$y_1(t) = e^{(a+bi)t} = e^{at}e^{ibt} = e^{at}(\cos bt + i\sin bt)$$

$$y_2(t) = e^{(a-bi)t} = e^{at}e^{ibt} = e^{at}(\cos(-bt) + i\sin(-bt))$$

$$= e^{at}(\cos bt - i\sin bt)$$

Remark Since our ODE is linear & homogeneous, we can

- add two solutions
- multiply a solution by a scalar

.... and still have a solution.

Thus, $\frac{1}{2}(y_1(t) + y_2(t)) = e^{at}\cos bt$ is a solution

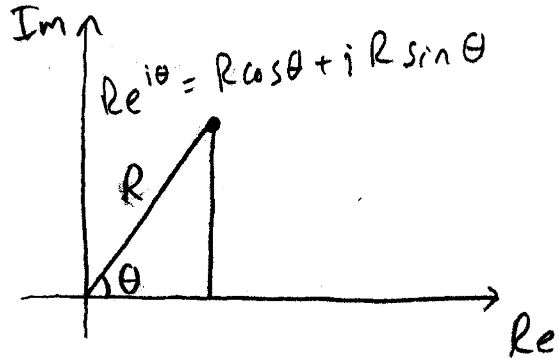
and $\frac{1}{2i}(y_1(t) - y_2(t)) = e^{at}\sin bt$ is a solution.

Conclusion: The general solution can be written as

$$y(t) = Ae^{at}\cos bt + Be^{at}\sin bt, \text{ or } \boxed{y(t) = e^{at}(A\cos bt + B\sin bt)}$$

Review: Basic complex numbers & Euler's formula.

In the complex plane, a point z at a dist. R from $\vec{O}$ & angle θ is



$$Re^{i\theta} = R\cos\theta + iR\sin\theta$$

From this, it is "easy" to see that

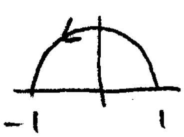
$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta.$$

($\theta \mapsto -\theta$ is a reflection across the x-axis. This preserves the x-coordinate but flips the sign of the y-coordinate).

Euler's formula is $\boxed{e^{it} = \cos t + i\sin t}$, and it implies some neat facts:

$$\left.\begin{aligned} e^{it} &= \cos t + i\sin t \\ e^{-it} &= \cos t - i\sin t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \tfrac{1}{2}(e^{it} + e^{-it}) &= \cos t \\ \tfrac{1}{2i}(e^{it} - e^{-it}) &= \sin t \end{aligned}}$$

Also, $\boxed{e^{i\pi} = -1}$! Visually,



$\theta = \pi$ radians.

Another way to see Euler's formula, from Taylor series:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad \cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \quad \sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \frac{(it)^8}{8!} + \dots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} +/- \dots$$

$$i\sin t = it - i\frac{t^3}{3!} + i\frac{t^5}{5!} - i\frac{t^7}{7!} +/- \dots$$

$$\Rightarrow e^{it} = \cos t + i\sin t$$

Inhomogeneous linear 2^{nd} order ODE's: $\boxed{y'' + p(t)y' + q(t)y = f(t)}$ ← forcing term

* Big idea #1: Suppose a homogeneous ODE has solutions $y_1(t)$ and $y_2(t)$. Then $C_1y_1(t) + C_2y_2(t)$ is a solution as well.

Proof: Plug $C_1y_1 + C_2y_2$ back into $y'' + py' + qy = 0$:

$$(C_1y_1 + C_2y_2)'' + p(t)(C_1y_1 + C_2y_2)' + q(t)(C_1y_1 + C_2y_2)$$

$$= (C_1y_1'' + p(t)C_1y_1' + q(t)C_1y_1) + (C_2y_2'' + p(t)C_2y_2' + q(t)C_2y_2)$$

$$= C_1(\underbrace{y_1'' + p(t)y_1' + q(t)y_1}_{=0}) + C_2(\underbrace{y_2'' + p(t)y_2' + q(t)y_2}_{=0}) = 0. \checkmark$$

* Big idea #2: The general solution of a linear ODE has the form $y(t) = y_h(t) + y_p(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$, where $y_p(t)$ is any particular solution, and $y_h(t)$ solves the related "homogeneous equation," $y'' + p(t)y' + q(t)y = 0$.

Proof: Consider $y(t) - y_p(t)$; the general sol'n minus $y_p(t)$:

$$\begin{array}{r} y'' + py' + qy = f \\ -(y_p'' + py_p' + qy_p = f) \\ \hline (y-y_p)'' + p\cdot(y-y_p)' + q\cdot(y-y_p) = 0 \end{array}$$

$\Rightarrow$ $y - y_p$ solves the homog eq'n

i.e, $y - y_p = y_h = C_1y_1 + C_2y_2$

$\Rightarrow$ $y = C_1y_1 + C_2y_2 + y_p$. ✓

How to find a particular sol'n: "Method of undetermined coefficients."

Technique: Guess! (We'll see how to guess right.)

Example 1: $y'' - 5y' + 4y = e^{3t}$. (Homog. eq'n: $y'' - 5y' + 4y = 0$)

First step: Solve the homog. eq'n: $y_h(t) = C_1 e^{4t} + C_2 e^{t}$.

Next, <u>guess</u> that there will be a sol'n $y_p(t) = a e^{3t}$ (why?)

Plug this back in (to solve for a). <u>Note</u>: $y_p' = 3ae^{3t}$, $y_p'' = 9ae^{3t}$

$$(9y'' \quad - \quad 5y' \quad + \quad 4y \quad = e^{3t})$$

$$9ae^{3t} - 5(3ae^{3t}) + 4ae^{3t} = e^{3t}$$

Combine terms: $\quad -2ae^{3t} = e^{3t} \implies a = -\frac{1}{2}$.

Thus, $y_p(t) = -\frac{1}{2}e^{3t}$ is a solution!

Using "Big idea # 2", $\quad y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^{4t} + C_2 e^{t} - \frac{1}{2}e^{3t}}$

<u>Think</u>: Why did this work?

<u>Ans</u>: Because the forcing term $f(t)$ and its derivatives had the "same form" (so we could get them to cancel out.)

<u>Example 2</u>: $y'' + 2y' - 3y = 5\sin 3t$. (Note: $y_h(t) = C_1 e^{t} + C_2 e^{-3t}$)

<u>Problem</u>: If we try $y_p(t) = a\sin 3t$, then $y_p(t) = 3a\cos 3t$ ...

Not of the 'same form.' (So they won't cancel)

<u>Fix</u>: Consider a more general particular solution:

$$y_p(t) = a\cos 3t + b\sin 3t$$

$$y_p'(t) = -3a\sin 3t + 3b\cos 3t$$

$$y_p''(t) = -9a\cos 3t - 9b\sin 3t$$

These all "have the same form"

Plug back in:

$$y_p'' + 2y_p' - 3y_p = (-9a\cos 3t - 9b\sin 3t) + (-6a\sin 3t + 6b\cos 3t) - (3a\cos 3t + 3b\sin 3t)$$

$$= \underbrace{(-12a+6b)}_{=0}\cos 3t + \underbrace{(-6a-12b)}_{=5}\sin 3t = \underbrace{5\sin 3t}_{f(t)}$$

Thus, we have $\left.\begin{array}{l} -12a+6b=0 \\ -6a-12b=5 \end{array}\right\} \Rightarrow a = -\frac{1}{6},\ b = -\frac{1}{3}$

$$\Rightarrow y_p(t) = -\tfrac{1}{6}\cos 3t - \tfrac{1}{3}\sin 3t$$

The general solution is therefore

$$y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^{t} + C_2 e^{-3t} + \left(-\tfrac{1}{6}\cos 3t - \tfrac{1}{3}\sin 3t\right)}$$

Example 3: (polynomial forcing term)

$y'' + 2y' - 3y = 6t^2 + t - 2$ Again, $y_h(t) = C_1 e^{t} + C_2 e^{-3t}$

Assume there's a particular sol'n of the form $\boxed{y_p(t) = at^2 + bt + C}$

Why?? (b/c $y_p'' + 2y_p' - 3y_p$ will also be a degree-2 poly.)

So, all we have to do is find a, b, c.

Plug back in: $y_p' = 2at + b$, $y_p'' = 2a$.

$$y'' + 2y' - 3y = (2a) + 2(2at+b) - 3(at^2+bt+c) = 6t^2 + t - 2$$

$$\underbrace{(-3a)}_{=6}t^2 + \underbrace{(4a-3b)}_{=1}t + \underbrace{(2a+2b-3c)}_{=-2} = 6t^2 + t - 2$$

$$\begin{cases} -3a = 6 \\ 4a - 3b = 1 \\ 2a + 2b - 3c = -2 \end{cases} \Rightarrow \begin{array}{l} a = -2 \\ b = -3 \\ c = -\frac{8}{3} \end{array} \Rightarrow \begin{array}{l} y_p(t) = -2t^2 - 3t - \frac{8}{3} \\ y(t) = y_h(t) + y_p(t) \\ \boxed{y(t) = C_1 e^{t} + C_2 e^{-3t} - 2t^2 - 3t - \frac{8}{3}} \end{array}$$

What could go wrong with this method?

What if the forcing term is a solution to the homogeneous eq'n?

Example 4: $y''-3y'+2y=e^{2t}$.

"Characteristic eq'n": $r^2-3r+2=(r-1)(r-2)$

$\Rightarrow y_h(t)=C_1e^t+C_2e^{2t}$.

Assume there's a particular sol'n of the form

$$y_p(t)=ae^{2t}, \quad \text{so} \quad y_p'=2ae^{2t}, \quad y_p''=4ae^{2t}.$$

Plug back in: $(4ae^{2t})-3(2ae^{2t})+2(ae^{2t})=e^{2t}$

$\Rightarrow 0ae^{2t}=e^{2t}$ no solution for a!

What happened?

Note: ae^{2t} solves $y''-3y'+2y=0$ (homog. eq'n),

thus it will <u>never</u> solve $y''-3y'+2y=e^{2t}\neq 0$.

To "Fix" this, assume instead that $\boxed{y_p(t)=ate^{2t}}$

$$y_p'(t)=2ate^{2t}+ae^{2t}, \quad y_p''(t)=4ate^{2t}+4ae^{2t}$$

Plug back in: $(4ate^{2t}+4ae^{2t})-3(2ate^{2t}+ae^{2t})+2(ate^{2t})=e^{2t}$

$\Rightarrow 0ate^{2t}+ae^{2t}=e^{2t}$

$\Rightarrow a=1 \Rightarrow y_p(t)=te^{2t}$

Thus, the general sol'n is $y(t)=y_h(t)+y_p(t)$

$$\boxed{y(t)=C_1e^t+C_2e^{2t}+te^{2t}}$$

Example 5: $y'' + 2y' - 3y = 5\sin 3t + 6t^2 + t - 2$ $(*)$

Again, $y_h(t) = C_1 e^{t} + C_2 e^{-3t}$

Recall that $-\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$ solves $y'' + 2y' - 3y = 5\sin 3t$ (Ex. 2)

and $-2t^2 - 3t - \frac{8}{3}$ solves $y'' + 2y' - 3y = 6t^2 + t - 2$ (Ex. 3)

Convince yourself that $y_p(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - 2t^2 - 3t - \frac{8}{3}$ solves $(*)$.

Thus, the general solution is $y(t) = y_h(t) + y_p(t)$, i.e.,

$$\boxed{y(t) = C_1 e^{t} + C_2 e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - 2t^2 - 3t - \frac{8}{3}}$$

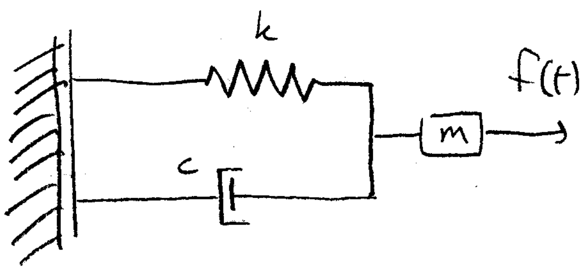
In general (combination forcing terms)

* Suppose $y'' + py' + qy = f(t)$ has sol'n $y_f(t)$

and $y'' + py' + qy = g(t)$ has sol'n $y_g(t)$.

Then $y'' + py' + qy = \alpha f(t) + \beta g(t)$ has sol'n $\alpha y_f(t) + \beta y_g(t)$.

Application: Harmonic motion.



Recall mass-spring systems.

Let $x(t)$ = displacement of the mass

Then $x(t)$ satisfies the following ODE: $\boxed{m x'' + 2c x' + \omega_0^2 x = f(t)}$

c = damping constant

ω_0 = frequency

$f(t)$ = driving force.

Example 1: Simple harmonic motion (no damping / driving force)

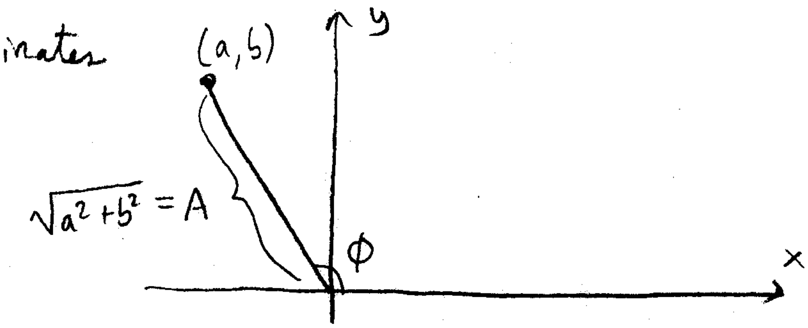
$x'' + kx = 0$, $k = \omega^2 > 0$ (Here, ω will be "frequency").

$\boxed{x'' = -\omega^2 x}$ $\Rightarrow$ $\boxed{x(t) = a\cos\omega t + b\sin\omega t}$

What does this function "look like"?

Here's a trick: We can actually write it as a single cosine wave!

Let's switch to polar coordinates



$*\ (a,b) = (A\cos\phi,\ A\sin\phi$

Sneaky little trick:

$$x(t) = \underbrace{a}\cos(\omega t) + \underbrace{b}\sin\omega t \qquad A \geq 0, \qquad -\pi < \phi \leq \pi$$

$$= \underbrace{A\cos\phi}\cos(\omega t) + \underbrace{A\sin\phi}\sin(\omega t)$$

$$= A\cos(\phi - \omega_0 t) \qquad \text{Trig identity: } \cos(x-y) = \cos x\cos y + \sin x \sin y$$

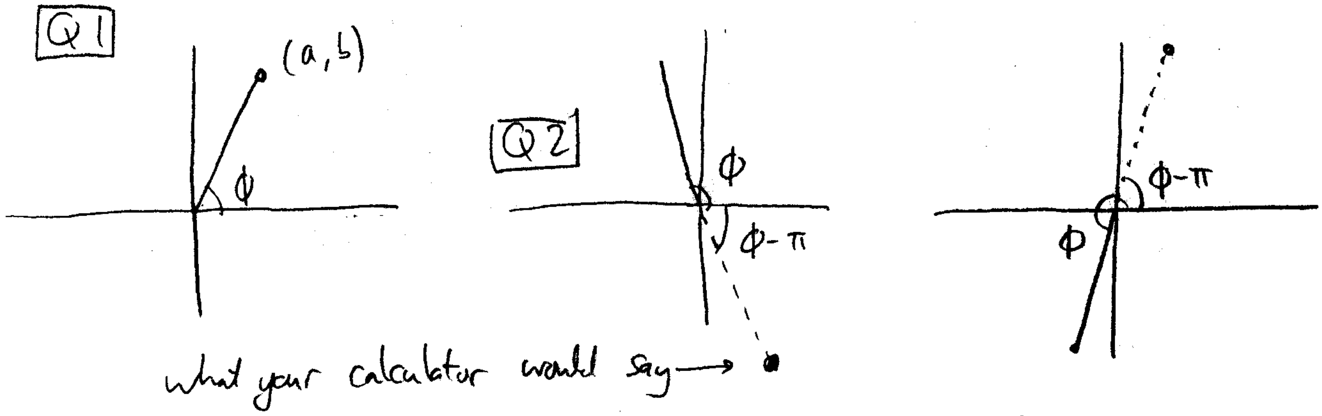
$$= A\cos(\omega t - \phi)$$

Big idea: Any function $x(t) = a\cos(\omega t) + b\sin(\omega t)$ can be written as a single cosine wave, with

* Amplitude $A = \sqrt{a^2+b^2}$
* Phase shift $\frac{\phi}{\omega}$, where "$\phi = \tan^{-1}(b/a)$"

So, $x(t) = A\cos(\omega t - \phi) = \boxed{A\cos\left(\omega\left(t - \frac{\phi}{\omega}\right)\right)}$

Note: Since $\underbrace{-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}}_{\text{for calculators}}$, $\quad \phi = \begin{cases} \arctan(b/a) & Q\,1,4 \\ \arctan(b/a) + \pi & Q\,2 \\ \arctan(b/a) - \pi & Q\,3 \end{cases}$



Example: $\quad x(t) = -3\cos t + 4\sin t$

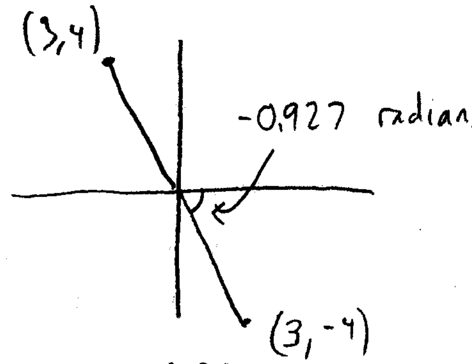
$$A = \sqrt{3^2 + 4^2} = 5$$

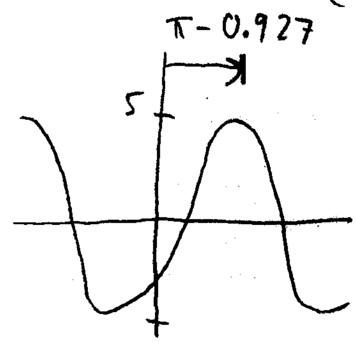
$$\arctan(-4/3) = -0.927$$

according to your calculator

So, $\phi = -0.927 + \pi$

$\Rightarrow \; x(t) = 5\cos\left[t - (-0.927 + \pi)\right]$





Example 2: Simple harmonic motion + external force (grav.)

A 2 kg mass is suspended from a spring. The displacement of the spring once the mass is attached is 0.5 m. If the mass is displaced 0.12 m downward from equilibrium, set up and solve the initial value problem that models this.

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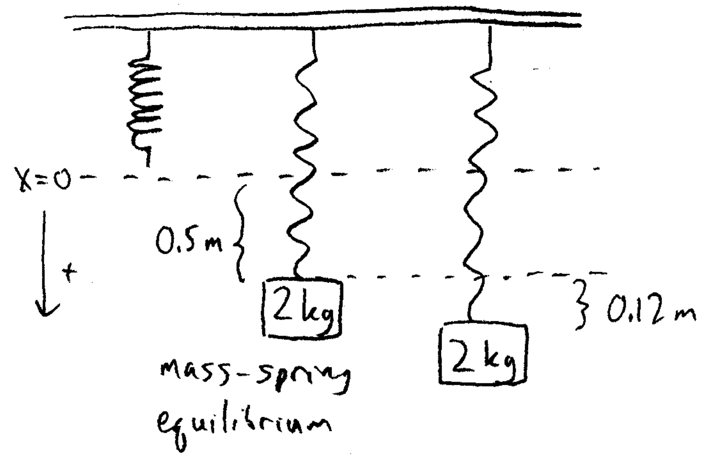


- 1st, determine the spring constant: $kx_0 = mg$.
(at equilibrium, spring force = grav. force).

$\Rightarrow k = \frac{mg}{x_0} = \frac{2 \cdot 9.8}{0.5} = 39.2 \text{ N/m}$

- 2nd: $F = mx'' = \Sigma$ forces

$$\underbrace{mx''}_{\text{total force acting on the weight}} = \underbrace{-\mu x'}_{\substack{\text{damping} \\ (=0)}} \underbrace{- kx}_{\text{spring}} + \underbrace{mg}_{\text{grav.}}$$

$mx'' = -kx + mg$

$2x'' = -kx + kx_0 = -k(x - x_0) = -39.2\,(x - 0.5)$

$$\boxed{2x'' + 39.2(x - 0.5) = 0, \quad x(0) = 0.62, \quad x'(0) = 0}$$

Let's solve this: $2x'' + 39.2x = 19.6$

$x_h(t) = A \cos \omega t + B \sin \omega t$ where $\omega = \sqrt{19.6}$

$x_p(t) = 0.5$ (set $x'' = 0$ & solve for x).

General solution: $x(t) = A \cos \omega t + B \sin \omega t + 0.5$

Plug in $x(0) = 0.62$ & $x'(0) = 0$ ← simplier IC. Use it first.

$x'(t) = -A\omega \sin \omega t + B\omega \cos \omega t$

$x'(0) = 0 + B\omega = 0 \Rightarrow B = 0$

$x(t) = A \cos \omega t.$

$x(0) = A = 0.62 \implies \boxed{x(t) = 0.12 \cos(\sqrt{19.6}\, t) + 0.5}$

Example 3: Damped harmonic motion. $(c \neq 0)$

$$x'' + 2cx' + \omega_0^2 x = 0, \qquad c > 0$$

Assume $x(t) = e^{rt} \implies r^2 + 2cr + \omega_0^2 = 0$

$\implies r = -c \pm \sqrt{c^2 - \omega_0^2}$

3 cases:

(i) Complex roots $(c < \omega_0)$: underdamped.

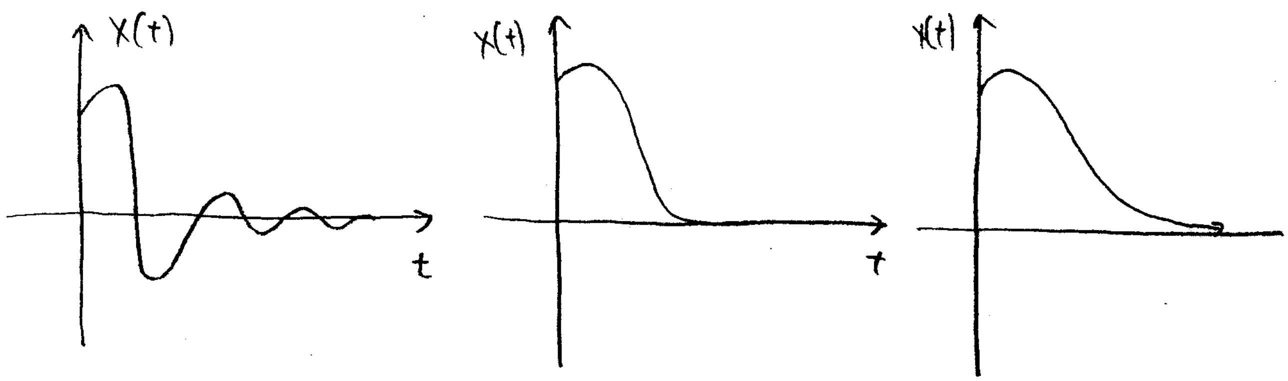
$$x(t) = e^{-ct}(a \cos \omega_0 t + b \sin \omega_0 t)$$

(ii) Double root $(c = 0)$: critically damped

$$x(t) = C_1 e^{rt} + C_2 t e^{rt} \qquad (\text{note: } r_1 = r_2 < 0)$$

(iii) 2 real roots $(c > \omega_0)$ overdamped.

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$



underdamped　　critically damped　　overdamped

Example 4: Forced harmonic motion; $f(t) \neq 0$.

e.g.,
- Spring attached to a motor
- Source voltage is sinuosidal

$$x'' + 2cx' + \omega_0 x = A \cos \omega t$$

damping coefficients; natural frequency; driving frequence

Simplest case: No damping $(c=0)$

$$x'' + \omega_0^2 x = A \cos \omega t$$

Homog. eq'n: $x_h'' + \omega_0^2 x = 0$

$$x_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

Case 1: $\omega \neq \omega_0$.

$x_p(t) = a \cos \omega t + b \sin \omega t$. Need to solve for a & b.

Plug x_p back in:

$$x_p'' + \omega_0^2 x_p = \overbrace{a(\omega_0^2 - \omega^2)}^{=A} \cos \omega t + \overbrace{b(\omega_0^2 - \omega^2)}^{=0} \sin \omega t$$

$$= A \cos \omega t + 0 \sin \omega t$$

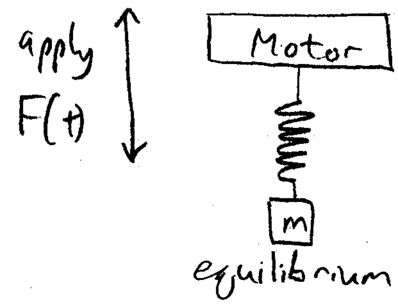
$$\Rightarrow \left.\begin{array}{l} a(\omega_0^2 - \omega^2) = A \\ b(\omega_0^2 - \omega^2) = 0 \end{array}\right\} \Rightarrow a = \frac{A}{\omega_0^2 - \omega^2}, \quad b = 0,$$

i.e, $x_p(t) = \dfrac{A}{\omega_0^2 - \omega^2} \cos \omega t$ (Note: $A \to \infty$ as $\omega_0 \to \omega$!)

General sol'n: $x(t) = x_h(t) + x_p(t) = \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \dfrac{A}{\omega_0^2 - \omega^2} \cos \omega t}$

What does this solution look like?

First, consider equilibrium: $\left.\begin{array}{l} x(0)=0 \\ x'(0)=0 \end{array}\right\}$
(for simplicity)



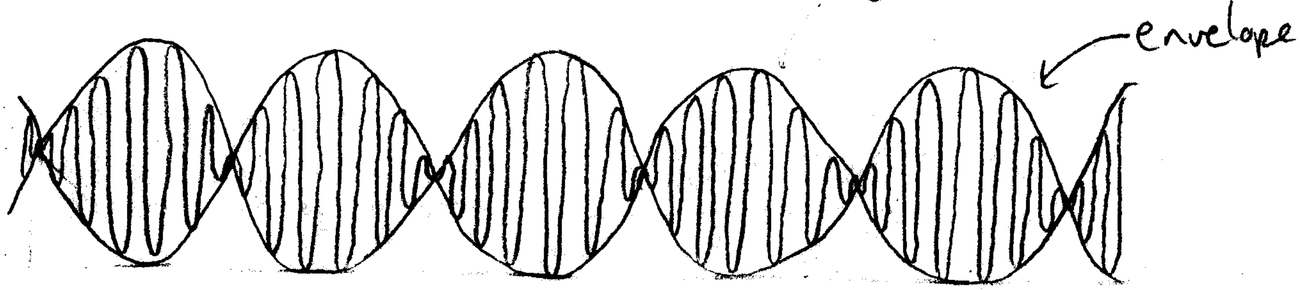
$C_2 = 0, \quad C_1 = \dfrac{-A}{\omega_0^2 - \omega^2}$

$$\Rightarrow \quad x(t) = \frac{A}{\omega_0^2 - \omega^2}\left(\cos \omega t - \cos \omega_0 t\right)$$

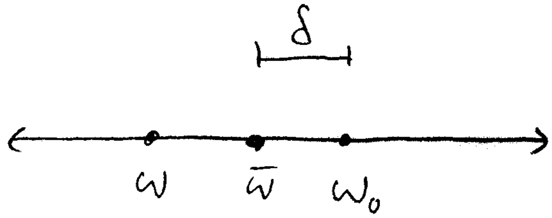
- Superposition of waves with different frequencies.

Has anyone experienced this in real life? (Think music!)

Example: $x(t) = \cos(11t) - \cos(12t)$ (rough sketch)



How to quantify this?



Let $\bar{\omega} = \dfrac{\omega + \omega_0}{2}$ (ave. freq.)

and say $\omega = \bar{\omega} - \delta$

$\omega_0 = \bar{\omega} + \delta$

$$\Rightarrow \quad x(t) = \frac{A}{\omega_0^2 - \omega^2}\left(\cos \omega t - \cos \omega_0 t\right) = \underbrace{\left(\frac{A \sin \delta t}{2\bar{\omega}\delta}\right)}_{\text{Amplitude is sinusoidal}} \sin \bar{\omega} t$$

use trig identities

<u>Case 2</u>: $\omega = \omega_0$ (Recall: $f(t) = \cos \omega t$ is the forcing term)

The ODE is now: $x'' + \omega_0 x'' = A \cos \omega_0 t$.

But $x_p(t) = A \cos \omega_0 t$ solves the homog eq'n (the "problem case").

So we must try $x_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t)$

Plug x_p back in:

$$x_p'' + \omega_0^2 x_p = \left[2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) + t\omega_0^2(-a \cos \omega_0 t + b \sin \omega_0 t)\right]$$
$$+ \omega_0 t(a \cos \omega_0 t + b \sin \omega_0 t)$$
$$= 2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) = \underbrace{A \cos \omega t + 0 \sin \omega t}_{\text{forcing term}}$$

$$\left.\begin{array}{l} -2\omega_0 a = 0 \\ 2\omega_0 b = A \end{array}\right\} \Rightarrow a = 0, \quad b = \frac{A}{2\omega_0}$$

Thus $x_p(t) = \dfrac{A}{2\omega_0} t \sin \omega_0 t$

General sol'n: $x(t) = x_h(t) + x_p(t)$

$$= \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \overbrace{\left(\frac{A}{2\omega_0} t\right)}^{\text{Amplitude} \to \infty!} \sin \omega_0 t}$$

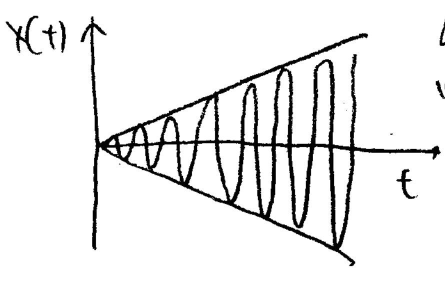
Look at the long-term behavior. This wave "blows up"!

Example: Again, consider starting at equilibrium: $x(0) = 0$, $x'(0) = 0$.

$x(0) = C_1 = 0$, $\quad x'(t) = C_2 \omega_0 \cos \omega_0 t + \frac{A}{2} t \cos \omega_0 t + \frac{A}{2\omega_0} \sin \omega_0 t$.

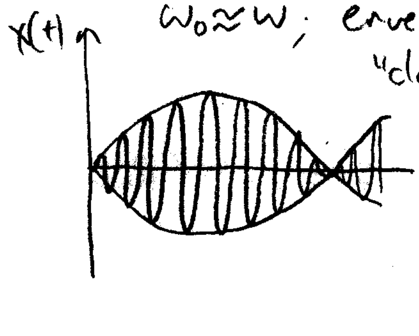
$x'(0) = C_2 = 0 \Rightarrow \boxed{x(t) = \dfrac{A}{2\omega_0} t \sin \omega_0 t}$

Real-life example: Tacoma Narrows Bridge



Envelope never "closes up" (like dividing by 0).

vs.



$\omega_0 \approx \omega$; envelope "closes up."

2nd order, non-constant coefficient ODE's:

Consider the following: $x^2y'' + xy' - y = 0$, Solve for $y(x)$.

What's a good guess? Try $y(x) = x^r$. (Why?)

$$y'(x) = r x^{r-1}, \quad y''(x) = r(r-1)x^{r-2}.$$

Plug back in: $x^2y'' + xy' - y = x^2 r(r-1)x^{r-2} + x r x^{r-1} - x^r = 0$

$$x^r(r^2-1) = 0 \implies r = \pm 1.$$

we have 2 solutions: $y_1(x) = x$, $y_2(x) = x^{-1}$.

$$\implies \boxed{y(x) = C_1 x + C_2 x^{-1}}$$

What if r is complex? Consider $x^2y'' + xy' + y = 0$.

Again, guess $y(x) = x^r$. $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

Plug back in... get $x^r(r^2+1) = 0 \implies r = \pm i$.

Thus $y_1(x) = x^i$, $y_2(x) = x^{-i}$ are solutions.

Simplify these: $y_1(x) = x^i = (e^{\ln x})^i = e^{i\ln x} = \cos(\ln x) + i\sin(\ln x)$

$y_2(x) = x^{-i} = (e^{\ln x})^{-i} = e^{-i\ln x} = \cos(\ln x) - i\sin(\ln x)$

$$\left.\begin{aligned} \tfrac{1}{2}(y_1+y_2) &= \cos(\ln x) \\ \tfrac{1}{2i}(y_1-y_2) &= \sin(\ln x) \end{aligned}\right\} \text{Distinct solutions!}$$

Thus, our general sol'n is $\boxed{y(x) = C_1\cos(\ln x) + C_2\sin(\ln x).}$

Remark: In the more general case when $r = a \pm bi$, the solution is $\boxed{y(x) = C_1 e^{at}\cos(b \ln x) + C_2 e^{at}\sin(b \ln x)}$

Remark: If r is a repeated root, then we must assume that $y(x) = v(x)\, x^r$, and we'll get $\boxed{y(x) = C_1 x^r + C_2 x^r \ln x}$ (work omitted, it's slightly tedious).

Let's make things harder.

Consider $y'' - 4x y' + 12 y = 0$.

What do we assume the solution will be?

Note: $y(x) = x^r$ won't work!

Because if $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$,

then $y'' - 4xy' + 12y = r(r-1)\underline{x^{r-2}} + 4r\underline{x^r} + 12\underline{x^r} = 0$

Maybe try $y(x) = a x^r + b x^{r-2}$?

Then, we'll get $(\quad\quad)x^{r-4} + (\quad\quad)x^{r-2} + (\quad\quad)x^r = 0$.

This will give us a solution, since we have 3 equations (set coeffs to 0) and 3 unknowns (a, b, r).

But, it'll only give us one solution (up to scalars).

<u>Better method</u>: Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Why? Because most "nice" functions have a Taylor series expansion, so let's find that.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

<u>Plug back in</u>: $\underbrace{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}}_{\text{re-write this so we can combine terms.}} - 4\sum_{n=0}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0 \quad (*)$

Let $m = n-2$: $\displaystyle\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} x^m$

(so $n = m+2$)

$$(\text{why?}) \quad = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m.$$

We've <u>shifted</u> the indices without changing the series.

* Observe (example, for motivation).

If $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

then $f'(x) = \boxed{\sum_{n=0}^{\infty} n x^{n-1}} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \boxed{\sum_{n=0}^{\infty} (n+1) x^n}$

Now, switch back to using n (from m):

$(*)$ becomes $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4 \sum_{n=0}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1) a_{n+2} + (12 - 4n) a_n}_{\text{set } = 0} \Big] x^n = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + (12-4n)\,a_n = 0 \quad \text{for } \underline{\text{all}}\ n.$$

$$\Rightarrow \boxed{a_{n+2} = \frac{4(n-3)}{(n+2)(n+1)}\,a_n}.$$ This is a <u>recurrence relation</u>.

<u>Note</u>: $y(0) = a_0$ and $y'(0) = a_1$.

Choose <u>any</u> a_0. <u>All</u> the even a_n's are determined.

Choose <u>any</u> a_1. <u>All</u> the odd a_n's are determined.

Thus, we have a 2-parameter infinite family of solutions, and so $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is the <u>general</u> solution, given this recurrence.

<u>Remark</u>: Since the odd & even terms are independent of each other, we could write $y_1(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$, $y_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $y(t) = C_1 y_1(x) + C_2 y_2(x)$.

Let's compute the first few terms (in terms of a_0 & a_1).

$a_2 = \frac{-12}{2} a_0 = -6a_0$ $\qquad$ $a_3 = \frac{-8}{3!} a_1 = \frac{-4}{3} a_1$

$a_4 = \frac{-4}{4\cdot 3} a_2 = \frac{(-4)(-12)}{4!} a_0 = 2a_0$ $\qquad$ $a_5 = 0$

$a_6 = \frac{(4)(-4)(-12)}{6!} a_0 = \frac{4}{15} a_0$ $\qquad$ $a_7 = 0$

$\vdots$

and so on... $a_n = \dfrac{(4\cdot 0-12)(4\cdot 2-12)(4\cdot 4-12)\cdots\boxed{4(n-2)-12}}{n!}$ $\quad (\ast\ast)$

Remark: If $a_0 = 0$, then $y(x) = a_1 x + a_3 x^3$

$$= a_1 x - \frac{4}{3} a_1 x^3 = \boxed{a_1 \left(x - \frac{4}{3} x^3 \right)}$$

This is the only polynomial solution, up to scalars (why?).

Summary: To solve $y'' - 4xy' + 12y = 0$, we

- Assumed $y(x) = \sum_{n=0}^{\infty} a_n x^n$
- Plugged $y(x)$ back into the ODE.
- Combined into a single sum $\sum_{n=0}^{\infty} [\quad] x^n = 0$

$\Big\}$ shifting of indices required

- Set coefficients equal to 0 to get a recurrence: $a_{n+2} = (\quad) a_n$.

Quick review of power series:

Def: A power series centered at x_0 is a series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n = \lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} a_n (x - x_0)^n}_{\text{"partial sum"}}$

* Henceforth, we will only consider power series centered at $x_0 = 0$, i.e, $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

A power series converges at x if the sequence of partial sums converges. Otherwise, it diverges.

Example: $\lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} x^n = e^x$ for all x.

Non-example: $\lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n x^n$ diverges for $x=1$, because the sequence of partial sums of $\sum_{n=0}^{N} (-1)^n x^n = 1-1+1-1+/-\ldots$ is $1, 0, 1, 0, 1, 0, \ldots$

Key point: Sometimes a series won't converge everywhere.

Example: $y(x) = \sum_{n=0}^{\infty} x^n$:
- Converges to $\frac{1}{1-x}$ if $|x|<1$
- Diverges if $|x| \geq 1$.

Def: The radius of convergence is the largest number R such that if $|x-x_0| < R$, $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges. If it converges for all x, we say $R=\infty$.

Examples: $y(x) = \sum_{n=0}^{\infty} x^n$ has $R=1$

$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has $R=\infty$. (it converges to e^x).

The Ratio Test, for computing R.

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}, \text{ if this limit exists.}$$

Example 1: Taylor series for $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n = x - x^2 + x^3 - x^4 +/- \ldots$

So, $|a_n| = 1$ for all $n \geq 1 \Rightarrow R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \frac{1}{1} \Rightarrow \boxed{R=1}$

Example 2: $y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$. $a_n = \frac{1}{3^n}$, $\lim_{n\to\infty} |1/3^n| / |1/3^{n+1}| = 3 \Rightarrow \boxed{R=3}$

Example 3: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $a_n = \frac{1}{n!}$ $\lim_{n\to\infty} \frac{|1/n!|}{|1/(n+1)!|} = n+1 \to \infty \Rightarrow \boxed{R=\infty}$

Regular vs. singular points of ODEs:

Def: A function $f(x)$ is real analytic at x_0 if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \text{for some } R > 0.$$

i.e., real analytic $\Leftrightarrow$ has a power series.

Def: Consider the ODE $y'' + P(x)\, y' + Q(x)\, y = 0$.

* The point x_0 is an ordinary point if $P(x)$ & $Q(x)$ are real analytic at x_0.
* If x_0 is not ordinary, then it is a singular point.
  - If x_0 is singular, then it is regular if $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are real analytic at x_0

Remark: In most cases, "real analytic" just means "defined"

e.g., $\frac{1}{x}$ is not real analytic at $x_0 = 0$.

Why we care:

Theorem of Frobenius: Consider an ODE $y'' + P(x)\, y' + Q(x)\, y = \underbrace{f(x)}_{\text{(real analytic at } x_0)}$.

- If x_0 is an ordinary point, and P, Q, f have radii of convergence R_P, R_Q, R_f, respectively, then there is a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, with $R = \min\{R_P, R_Q, R_f\}$.

- If x_0 is a <u>regular singular point</u>, and $(x-x_0)P(x)$, $(x-x_0)^2Q(x)$, and $f(x)$ have radii of convergence R_P, R_Q, R_f, resp., then there is a <u>generalized power series solution</u> $y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$, for some constant r (possibly a fraction, or even complex).

<u>Note</u>: If x_0 is an irregular singular point, we're out of luck.

<u>Example</u>: Consider $y'' + x^2 y - 4y = 0$. Thus, $P(x) = x^2$, $Q(x) = -4$. $P(x)$ & $Q(x)$ are real analytic for all x_0, with radii of conv. $R=\infty$. Thus, by Frobenius, there is a solution $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, valid for all x (i.e, $R=\infty$).

<u>Example</u>: $y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0$, ($p$ is a parameter).

Here, $P(x) = \frac{-x}{1-x}$, $Q(x) = \frac{1}{1-x^2}$.

<u>Note</u>: $P(x)$ and $Q(x)$ are real analytic at $x=0$:

$$Q(x) = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots$$

$$P(x) = \frac{-x}{1-x^2} = -x \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} -x^{2n+1} = -x - x^3 - x^5 - x^7 - \cdots$$

By the ratio test, $R_P = R_Q = 1$.

Thus, by Frobenius, there is a solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $R=1$. (This ODE is called Chebyshev's equation).

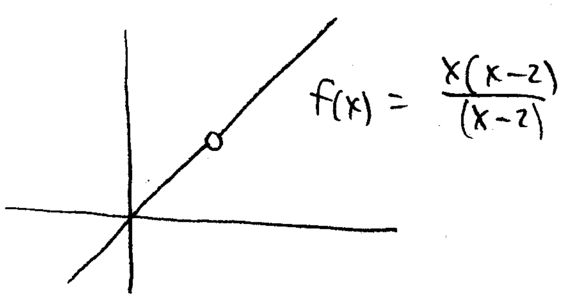
Example: $x^5 y'' + y' + y = 0.$

Write as $y'' + \frac{1}{x^5} y' + \frac{1}{x^5} y = 0.$ $\quad P(x) = \frac{1}{x^5}, \quad Q(x) = \frac{1}{x^5}.$

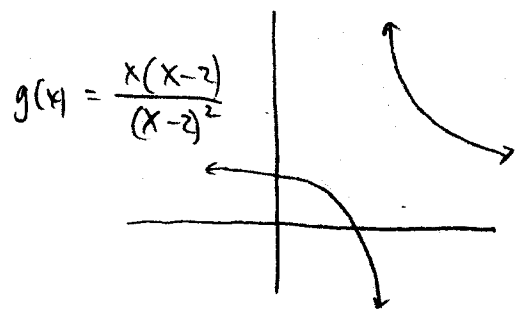
$x_0 = 0$ is an irregular singular point, since $x P(x) = \frac{1}{x^4}$ isn't defined at $x_0 = 0$.

Frobenius does not guarantee a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. But we could find one of the form $\sum_{n=0}^{\infty} a_n (x-1)^n$ if we wanted to. (B/c $x_0 = 1$ is regular).

Analogy of regular vs. irregular.



This singularity is "fixable"



This singularity is "unfixable"

Example: Solve $2xy'' + y' + y = 0.$

Write as $y'' + P(x) y' + Q(x) y = 0, \quad P(x) = \frac{1}{2x}, \quad Q(x) = \frac{1}{2x}$

$x_0 = 0$ is a regular singular point, since $x P(x) = \frac{1}{2}$ and $x^2 Q(x) = \frac{1}{2} x$ are real analytic (i.e, defined).

By Frobenius, there is a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We'll find it the same way as before.

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1}, \qquad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2}$$

$$2xy''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-1}.$$

Plug back into the ODE:

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$= x^r \left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r)\, a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \right] = 0$$

Shift indices up by one (let $m = n-1$, or just do in your head).

$$= \cancel{x^r} \left[\sum_{n=-1}^{\infty} (2n+2r+1)(n+r+1)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \right] = 0.$$

one extra term!

$$= \underbrace{(2r-1)\, r\, a_0\, x^{-1}}_{\text{Set } = 0.} + \sum_{n=0}^{\infty} \underbrace{\left[(2n+2r+1)(n+r+1)\, a_{n+1} + a_n \right]}_{\text{Set } = 0} x^n = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$(2r-1)\, r = 0$ "indicial equation"

$$\Downarrow$$

$$\boxed{r = 0 \text{ or } r = \tfrac{1}{2}}$$

$$\boxed{a_{n+1} = \frac{-1}{(2n+2r+1)(n+r+1)}\, a_n}$$

"recurrence relation"

We now have two generalized power series solutions:

$\underline{r=0}$: $\quad y_0(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_{n+1} = \dfrac{-1}{(2n+1)(n+1)}\, a_n$

$\underline{r=\tfrac{1}{2}}$: $\quad y_{1/2}(x) = \sqrt{x} \sum_{n=0}^{\infty} a_n x^n, \qquad a_{n+1} = \dfrac{-1}{(2n+2)(n+3/2)}\, a_n .$

Note: This time, choosing a_0 determines every a_n, but we still have 2 distinct solutions.

The general solution is $y(x) = A\, y_0(x) + B\, y_{1/2}(x)$, where y_0, $y_{1/2}$ are as above.

* The power series method really does come up in practice!!!

- Hermite's diff. eq: $y'' - 2xy' + 2py = 0$.
  Used for modeling simple harmonic oscillators in quantum mech.
- Legendre's diff eq: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$.
  Used for modeling spherically symmetric potentials in theory of Newtonian gravitation, and in electricity & magnetism (E & M).
- Bessel's equation: $x^2y'' + xy' + (x^2-p^2)y = 0$.
  Used for analyzing vibrations of a circular drum.
- Chebyshev's equation: $(1-x^2)y'' - xy' + p^2y = 0$.