Lecture 2.4: Solving first order inhomogeneous differential equations

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Math 2080, Differential Equations

Linear differential equations

High school algebra

A linear equation has the form $f(x) = ax + b$.

Differential equations

- A (1st order) linear differential equation has the form $y' = a(t)y + f(t)$.
- A (1st order) homogeneous linear differential equation has the form $y' = a(t)y$.

Examples

- $y' = t^2 y + 5$
- $y' = ty^2 + 5$
- $y' = t \sin y$
- $y' = y \sin t$
- $y' = t^3 + 2t^2 + t + 1$

Solving homogeneous ODEs

Method 1: Integrating factor

First steps

1. Write the equation as $y'(t) - a(t)y(t) = f(t)$;
2. Multiply both sides by $e^{-\int a(t)\,dt}$, the "integrating factor."

A familiar example

Example 1

Solve $y' = 2y + t$ using the integrating factor method.

Some practice

Find the integrating factor

(a) $y' + 4y = t^2$

(b) $y' + (\sin t)y = 1$

Some more practice

Find the integrating factor

(c) $y' - 12t^5 y = t^3$

(d) $y' + \frac{1}{t} y = 1$

Method 2: Variation of parameters

Steps to solving $y'(t) + a(t)y(t) = f(t)$

1. Find the solution $y_h(t)$ to the the related "homogeneous equation"

$$y'(t) + a(t)y(t) = 0.$$

2. Assume the general solution is $y(t) = v(t)y_h(t)$, and plug this back to the ODE and solve for $v(t)$.

Remarks

- This works "equally well" as the integrating factor (IF) method.
- Variation of parameters has a built-in "check-point" that IF does not.
- Variation of parameters can be used to solve 2nd order ODEs, whereas IF does not generalize.

Method 2: Variation of parameters

Example

Solve the ODE $y' = 2y + t$ using the variation of parameters method.