Lecture 3.5: Damped and forced harmonic motion

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Introduction

Harmonic motion

Recall that if $x(t)$ is the displacement of a mass m on a spring, then $x(t)$ satsifies

$$mx'' + 2cx' + \omega_0^2 x = f(t),$$

where

- c is the damping constant
- ω_0 is frequency
- $f(t)$ is the external driving force

In this lecture, we will analyze the cases when $c \neq 0$ and when $f(t)$ is sinusoidal.

Damped harmonic motion

The homogeneous case

Divide through by the mass m and we get a 2nd order constant coefficient ODE:

$$x'' + 2cx' + \omega_0^2 x = 0$$

Forced harmonic motion: $f(t) \neq 0$

An example

When the driving frequency is sinusoidal, the ODE for $x(t)$ is

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t\,,$$

where

- c is the damping coefficient;
- ω_0 is the natural frequency;
- ω is the driving frequency.

In this lecture, we will analyze the case when $c = 0$.

Case 1: $\omega \neq \omega_0$.

Forced harmonic motion: $f(t) \neq 0$

Summary so far

The general solution to $x'' + \omega_0^2 x = A\cos\omega t$, $\omega \neq \omega_0$ is

$$x(t) = x_h(t) + x_p(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{A}{\omega_0^2 - \omega^2}\cos\omega t\,.$$

Case 2: $\omega = \omega_0$

We need to solve $x'' + \omega_0^2 x = A\cos\omega_0 t$.

Case 2: $\omega = \omega_0$

Summary so far

The general solution to $x'' + \omega_0^2 x = A\cos\omega_0 t$ is

$$x(t) = x_h(t) + x_p(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{At}{2\omega_0}\sin\omega_0 t\,.$$