Lecture 3.6: Variation of parameters

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Overview

Methods of solving 1st order ODEs

(i) separation of variables
(ii) integrating factor
(iii) undetermined coefficients
(iv) variation of parameters

Beyond 1st order ODEs

The variation of parameters method works for n^{th} order linear ODEs.

It helps us find a particular solution of the form:

- $y_p(t) = v(t)y_h(t)$ for 1st order ODEs,
- $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ for 2nd order ODEs.

A simple example

Example 1

Find a particular solution to $y'' + y = \tan t$.

Example 1 (cont.)

To find a particular solution to $y'' + y = \tan t$, we assumed $y_p = v_1 \cos t + v_2 \sin t$, and then we derived the following system

$$\begin{cases} v_1' \cos t + v_2' \sin t = 0 \\ -v_1' \sin t + v_2' \cos t = \tan t \end{cases}$$

The general case

Example 2

Find a particular solution to $y'' + a(t)y' + b(t)y = f(t)$.

Example 2 (cont.)

To find a particular solution to $y'' + a(t)y' + b(t)y = f(t)$, we assumed that $y_p = v_1 y_1 + v_2 y_2$, and then we derived the following system

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = f(t) \end{cases}$$

Summary

Variation of parameters for 2nd order ODEs

The ODE $y'' + a(t)y' + b(t)y = f(t)$ has a solution $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where

- $y_h(t) = C_1y_1(t) + C_2y_2(t)$ is the general solution to the homogeneous equation
- $v_1(t) = \displaystyle\int \frac{-y_2(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$
- $v_2(t) = \displaystyle\int \frac{y_1(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$.

The general solution is thus $y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$.