Lecture 4.5: Phase portraits with real eigenvalues

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Last time: distinct negative eigenvalues

Example 1a

Consider the following homogeneous system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
It is easily verified that the eigenvalues and eigenvectors of **A** are

$$\lambda_1 = -0.25, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \qquad\qquad \lambda_2 = -0.05, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Thus, the general solution is $\mathbf{x}(t) = C_1 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Eigenvalues of opposite sign

Example 1b

Consider the following homogeneous system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
It is eaily verified that the eigenvalues and eigenvectors are

$$\lambda_1 = 3, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad\qquad \lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus, the general solution is $\mathbf{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Distinct positive eigenvalues

Example 1c

Consider a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 1, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad\qquad \lambda_2 = 5, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus, the general solution is $\mathbf{x}(t) = C_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Summary

When $\lambda_1 \neq \lambda_2$ are real

If $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{A}$ has real eigenvalues $\lambda_1 \neq \lambda_2$, and eigenvectors v_1, v_2, then the general solution is $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$.

We can plot the phase portrait (x_2 vs. x_1) by first drawing the "eigenvector lines".

- If $\lambda_i > 0$, then the solutions move *away from* $(0,0)$ because $\lim_{t\to\infty} |Ce^{\lambda t}\mathbf{v}| = \infty$.
- If $\lambda_i < 0$, then the solutions move *torward* $(0,0)$ because $\lim_{t\to\infty} |Ce^{\lambda t}\mathbf{v}| = 0$.

The solution curves off these lines "bend" depending on how different λ_1 and λ_2 are.