Lecture 4.8: Stability of phase portraits

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Summary of phase portraits

Suppose $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\det \mathbf{A} \neq 0$

- $\lambda_1 > \lambda_2 > 0$
- $\lambda_1 < \lambda_2 < 0$
- $\lambda_1 < 0 < \lambda_2$
- $\lambda_1 = \lambda_2$
- $\lambda = a \pm bi$
  - $a > 0$
  - $a < 0$
  - $a = 0$

A 1-parameter family

Example

Consider the following homogeneous system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \alpha & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The whole picture

A 2-parameter family

Suppose the characteristic equation of **A** is

$$\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = \lambda^2 - p\lambda + q = 0$$

Then the eigenvalues of **A** are $\lambda = \dfrac{p \pm \sqrt{p^2 - 4q}}{2}$.

What if $\det \mathbf{A} = 0$?

Higher order systems

An example: the SIR model

Consider an epidemic that spreads through a population, where

- $S(t) = \#$ susceptible people at time t;
- $I(t) = \#$ infected people at time t;
- $R(t) = \#$ recovered people at time t.

Initially, there are N susceptible (uninfected) people.