Lecture 4.9: Variation of parameters for systems

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Variation of parameters for non-systems

Variation of parameters is a "last resort" method for finding a particular solution, $y_p(t)$.

First order ODE: $y' + p(t)y = f(t)$

(i) Solve $y_h' + p(t)y_h = 0$: get $y_h(t) = Cy_1(t) = Ce^{-\int p(t)\,dt}$.

(ii) Find a particular solution of the form

$$y_p(t) = v(t)y_1(t) = e^{-\int p(t)\,dt} \int f(t)e^{\int p(t)\,dt}dt.$$

Second order ODE: $y'' + p(t)y' + q(t)y = f(t)$

(i) Solve $y_h'' + p(t)y_h' + q(t)y_h = 0$: get $y_h(t) = C_1y_1(t) + C_2y_2(t)$.

(ii) Find a particular solution of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).$$

It turns out that

$$v_1(t) = \int \frac{-y_2(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \qquad v_2(t) = \int \frac{y_1(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

These methods *always work*, assuming that you can find $y_h(t)$, and evaluate the integrals.

Variation of parameters for systems

2×2 system: $\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)$

(i) Solve $\boldsymbol{x}_h' = \boldsymbol{A}\boldsymbol{x}_h$: get

$$\boldsymbol{x}_h(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) = C_1\begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} + C_2\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}_{X_h(t)\boldsymbol{C}}.$$

(ii) Find a particular solution of the form $\boldsymbol{x}_p(t) = X_h(t)\boldsymbol{v}(t)$:

$$\boldsymbol{x}_p(t) = v_1(t)\boldsymbol{x}_1(t) + v_2(t)\boldsymbol{x}_2(t) = v_1(t)\begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} + v_2(t)\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}}_{X_h(t)\boldsymbol{v}(t)}.$$

A specific example

Example 1

Solve the initial value problem $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 10\cos t \\ 2e^{-t} \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 10 \\ 4 \end{bmatrix}.$

A general example

Example 2

Solve $y'' + p(t)y' + q(t)y = f(t)$ by turning it into a 2×2 system first.

Summary

The variation of parameters method finds a particular solution of an ODE, of the form:

(i) $y_p(t) = v(t)y_1(t)$ (1st order)
(ii) $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ (2nd order)
(iii) $\mathbf{x}_p(t) = X_h(t)\mathbf{v}(t)$ ($n \times n$ system).

We saw here that $\mathbf{x}_p(t) = X_h(t)\mathbf{v}(t) = X_h(t) \int X_h^{-1}(t)\mathbf{b}(t)\,dt$.

A second order ODE $y'' + p(t)y' + q(t)y = f(t)$ can be written as a system by setting $x_1 = y$, $x_2 = y'$, to get

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

Remarks

- This is the *only* way we know how to find a particular solution of a non-automonous ODE, e.g., $\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$.
- This method also works if $A(t)$ is non-constant, assuming that we can actually find $\mathbf{x}_h(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$.
- Such a solution is only defined where the Wronskian

$$W[\mathbf{x}_1(t), \mathbf{x}_2(t)] := \det \begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix}, \quad or \quad W[y_1(t), y_2(t)] := \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

is non-zero.