Lecture 5.3: Discontinuous forcing terms

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Step functions

Definitions

- Interval function: $H_{ab}(t) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & b \leq t < \infty \end{cases}$
- Heavyside function: $H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$
- Shifted Heavyside function: $H_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$

Writing step functions using Heavyside functions

Example 1

$$f(t) = \begin{cases} 2t, & 0 \leq t < 1 \\ 2, & t \geq 1 \end{cases}$$

Example 2

$$f(t) = \begin{cases} 3, & 0 \leq t < 4 \\ -5, & 4 \leq t < 6 \\ e^{7-t}, & 6 \leq t < \infty \end{cases}$$

Shifted functions

Key property

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(t-c)H(t-c)\} = e^{-cs}F(s)$

Shifted functions: $\mathcal{L}\{f(t-c)H(t-c)\} = e^{-cs}F(s)$

Examples

- Compute $\mathcal{L}\{(t-3)^2 H(t-3)\}$.
- Compute $\mathcal{L}\{t^2 H(t-3)\}$.
- Compute $\mathcal{L}\{e^{t-1} H(t-1)\}$.
- Compute $\mathcal{L}\{e^{7-t} H(t-6)\}$.

Shifted functions: $\mathcal{L}\{f(t-c)H(t-c)\} = e^{-cs}F(s)$

Example (revisited)

Compute the Laplace transform of $f(t) = \begin{cases} 3, & 0 \le t < 4 \\ -5, & 4 \le t < 6 \\ e^{7-t}, & 6 \le t < \infty \end{cases}$.

Recall that $f(t) = 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6)$.

ODEs with piecewise forcing terms

Example

Solve the IVP: $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$, where $f(t) = \begin{cases} 2t, & 0 \leq t < 1 \\ 2, & t \geq 1 \end{cases}$