Lecture 7.8: The two-dimensional wave equation

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Overview

Modeling with the wave equation

Consider a vibrating square membrane of length L, where the edges are held fixed. If $u(x, y, t)$ is the (vertical) displacement, then u satisfies the following IVP/BVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,L,t) = u(L,x,t) = 0$$
$$u(x,y,0) = h_1(x,y), \qquad u_t(x,y,0) = h_2(x,y).$$

The functions $h_1(x, y)$ and $h_2(x, y)$ are initial displacement and velocity, respectively.

Finding the general solution

Example 3

Solve the following IVP/BVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,\pi,t) = u(\pi,x,t) = 0$$
$$u(x,y,0) = x(\pi - x)\, y(\pi - y), \qquad u_t(x,y,0) = 0.$$

Solving the resulting IVP

Example 3 (cont.)

The general solution to the following IVP/BVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,\pi,t) = u(\pi,x,t) = 0$$
$$u(x,y,0) = x(\pi - x)\,y(\pi - y), \qquad u_t(x,y,0) = 0.$$

is $u(x,y,t) = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin mx \, \sin ny \, \cos(c\sqrt{m^2+n^2}\,t)$.