Lecture 8.2: Linearization and steady-state analysis

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Math 2080, Differential Equations

Recall: Competitive Lotka–Volterra equations

Example 1

Consider the following model of two species competing for a limited food supply:

$$\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.75 - Y - .5X) \end{cases}$$

Nullclines

Definition

A nullcline (or isocline with $c = 0$) is a line or curve where $X' = 0$ or $Y' = 0$.

Example 1 (cont.)

Let's find the nullclines of our system:

$$\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.75 - Y - .5X) \end{cases}$$

Linearization

Example 1 (cont.)

The 4 steady-states of the following system

$$\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.75 - Y - .5X) \end{cases}$$

are $(X^*, Y^*) = (0,0)$, $(1,0)$, $(0,.75)$, $(.5,.5)$. Let's linearize at $(X^*, Y^*) = (0,0)$.

Linearization

Example 1 (cont.)

Let's linearize this system at a non-zero critical point (X^*, Y^*).

$$\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.75 - Y - .5X) \end{cases}$$

The first step is to change variables: let $P = X - X^*$ and $Q = Y - Y^*$.

Linearization

Example 1 (cont.)

At a general critical point (X^*, Y^*) we changed variables $(P, Q) = (X - X^*, Y - Y^*)$ to get:

$$\begin{bmatrix} P' \\ Q' \end{bmatrix} = \begin{bmatrix} 1 - 2X^* - Y^* & -X^* \\ -.5Y^* & .75 - 2Y^* - .5X^* \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} + \begin{bmatrix} \text{non-linear} \\ \text{terms} \end{bmatrix}$$

Recall that our steady-states are $(X^*, Y^*) = (0, 0)$, $(1, 0)$, $(0, .75)$, $(.5, .5)$.

Analyzing a nonlinear system

Example 2

Consider the following model of two species competing for a limited food supply:

$$\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.8 - .6Y - X) \end{cases}$$

It is easy to check that there are four steady-state solutions: $(0,0)$, $(1,0)$, $(0, \frac{4}{3})$, $(\frac{1}{2}, \frac{1}{2})$.

Four types of dynamics

Competitive Lotka–Volterra equations

Consider the following model of two species competing for a limited food supply:

$$\begin{cases} X' = X(\varepsilon_1 - \sigma_1 X - \alpha_1 Y) \\ Y' = Y(\varepsilon_2 - \sigma_2 Y - \alpha_2 X) \end{cases}$$

- X-nullclines: $X = 0$ and $\varepsilon_1 - \sigma_1 X - \alpha_1 Y = 0$.
- Y-nullclines: $Y = 0$ and $\varepsilon_2 - \sigma_2 Y - \alpha_2 X = 0$.

When might linearization fail?

Consider a 2×2 matrix with characteristic equation $\lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + \det \mathbf{A} = \lambda^2 - p\lambda + q = 0$.