Math 2080: Differential Equations
Worksheet 4.7: Phase portraits with repeated eigenvalues

NAME:

Consider the system of differential equations: $\begin{cases} x_1' = 4x_1 + x_2, & x_1(0) = -1 \\ x_2' = -1x_1 + 2x_2, & x_2(0) = 1 \end{cases}$

(a) Write this in matrix form, $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

(b) Knowing that $\boldsymbol{A}$ has a repeated eigenvalue, $\lambda_{1,2} = 3$, and only one eigenvector, $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, write down a solution $\boldsymbol{x}_1(t)$ to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$.

(c) To find a second solution, assume that $\boldsymbol{x}_2(t) = te^{3t}\boldsymbol{v} + e^{3t}\boldsymbol{w}$. Plug this back into $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ and equate coefficients (of te^{3t} and e^{3t}) to get a system of two equations, involving $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{A}$.

(d) Solve for $\boldsymbol{v}$ by inspection. Plug this back into the second equation and solve for $\boldsymbol{w}$ (it will involve a constant, C).

(e) Using what you got for $\boldsymbol{v}(t)$ and $\boldsymbol{w}(t)$, write down a solution $\boldsymbol{x}_2(t)$ that is not a scalar multiple of $\boldsymbol{x}_1$. (Pick the simplest value of C that works.)

(f) Write down the general solution, $\boldsymbol{x}(t)$. As $t \to \infty$, which of the three terms of $\boldsymbol{x}(t)$ "grows faster"?

(g) Sketch the phase portrait. To determine which way the curves "sprial", compute $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ at $\boldsymbol{x} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ and see if this velocity vector is pointing upwards or downwards.