Lecture 1.1: Basic set theory

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What is a set?

Almost everybody, regardless of mathematical background, has an intiutive idea of what a set is: a collection of objects, sometimes called elements.

Sets can be finite or infinite.

Examples of sets

1. Let S be the set consisting of 0 and 1. We write $S = \{0, 1\}$.
2. Let S be the set of words in the dictionary.
3. Let $S = \emptyset$, the "empty set".
4. Let $S = \{2, A, \text{cat}, \{0, 1\}\}$.

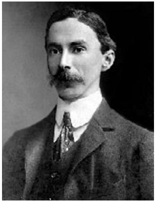
Repeated elements in sets are not allowed. In other words, $\{1, 2, 3, 3\} = \{1, 2, 3\}$. If we want to allow repeats, we can use a related object called a multiset.

Perhaps surprisingly, it is very difficult to formally define what a set is.

The problem is that some sets actually cannot exist!

For example, if we try to define the set of all sets, we will run into a problem called a paradox.

Russell's paradox



The following is called Russell's paradox, due to British philsopher, logician, and mathematician Bertrand Russell (1872–1970):

Suppose a town's barber shaves every man who doesn't shave himself.

Who shaves the barber?

Now, consider the set S of all sets which do not contain themselves.

Does S contain itself?

Later this class, we will encounter paradoxes that are not related to sets, but logical statements, such as:

This statement is false.

Set notation

We will usually denote a set by a capital letter.

If x is an element in A, we write $x \in A$. Otherwise, we write $x \notin A$.

Some commonly used sets

- $\mathbb{N}$: the natural numbers, $\{0, 1, 2, 3, \dots\}$
- $\mathbb{Z}$: the integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q}$: the rational numbers, $\{a/b \mid a, b \in \mathbb{Z},\ b \neq 0\}$.
- $\mathbb{R}$: the real numbers
- $\mathbb{C}$: the complex numbers, $\{a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$.

It should be clear what we mean by sets such as $\mathbb{Q}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$.

The vertical line, $|$, means "such that", or "where". We can also use a colon for this.

Commas are read as "and".

There are often multiple ways to describe a set, e.g.,

$$\{x \in \mathbb{R} \mid x^2 - 5x + 6 = 0\} = \{x \mid x \in \mathbb{R},\ x^2 - 5x = -6\} = \{2, 3\}.$$

Set notation

A set is finite if it has a finite number of elements. Otherwise, it is an infinite set.

The number of elements in a set A is called its cardinality, denoted $|A|$. If A is infinite, we may write $|A| = \infty$.

We will see later than there are different infinite cardinalities.

Definition

Let A and B be sets. We say that A is a subset of B if (and only if) every element of A is an element of B. We write this as $A \subseteq B$, or $B \supseteq A$.

Warning!

Unfortunately, the notations $A \subset B$ and $A \subseteq B$ mean the same thing.

If we want to say that there are elements in B that are not in A, we can write $A \subsetneq B$. We say A is a proper subset of B, or that B is strictly larger than A.

Remark

The term if and only if means "is equivalent to saying".

Example. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Basic set operations

Definition

The intersection of sets A and B is the set of elements in both A *and* B, denoted

$$A \cap B := \{x \mid x \in A \text{ and } x \in B\}.$$

Two sets are disjoint if they have no elements in common, i.e., if $A \cap B = \emptyset$.

The union of sets A and B is the set of elements in *either* A *or* B, denoted

$$A \cup B := \{x \mid x \in A \text{ or } x \in B\}.$$

Examples

1. If $A = \{2, 5, 8\}$ and $B = \{7, 5, 22\}$, then $A \cap B = \{5\}$ and $A \cup B = \{2, 5, 8, 7, 22\}$.
2. $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$, and $\mathbb{Z} \cap \mathbb{Q} = \mathbb{Z}$.
3. $A \cup \emptyset = A$ for any set A.

Set complements

Frequently, we will need to establish the set of all elements U under consideration, which we call the universe.

Definition

The complement of a set A is the set of all elements in U that are not in A:

$$A^c = \{x \in U \mid x \notin A\}.$$

Example

Let $A = \mathbb{N} = \{0, 1, 2, \dots\}$. What is the complement of A if the universe is:

(*i*) $U = \mathbb{Z}$, (*ii*) $U = \mathbb{Q}$, (*iii*) $U = \mathbb{R}$, (*iv*) $U = \mathbb{C}$, (*v*) $U = \mathbb{N}$.

Sometimes, the complement is denoted $\overline{A}$.

Relative complements

Definition

For sets A and B, the complement of A relative to B is the set of elements that are in B but not A:

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}.$$

The symmetric difference of A and B is the set of elements that are in one of these sets, but not the other:

$$A \oplus B = (A - B) \cup (B - A).$$

The complement of A relative to B can be denoted $A \setminus B$.

Exercises

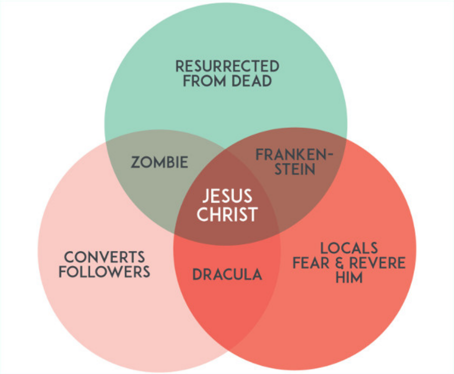
Compute $A - B$, $B - A$, and $A \oplus B$ in the following cases:

1. $A = \{1, 3, 8\}$ and $B = \{2, 4, 8\}$
2. Any set A, and $B = \emptyset$.
3. $A = \mathbb{R}$ and $B = \mathbb{Q}$.

Venn diagrams

A useful way to visualize a small number of sets and their intersections, unions, and relative complements, is with a Venn diagram.

Social media has caused these to become mainstream, though they are often used incorrectly.



Cartesian products

Definition

The Cartesian product of sets A and B is the set of ordered pairs:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Examples

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Then

- $A \times B =$
- $B \times A =$
- $A \times A =$

Similarly, we can define the Cartesian product of three (or more) sets. For example,

$$A \times B \times C = \{(a, b, c) \mid a \in A,\ b \in B,\ c \in C\}.$$

It is common to use exponents if the sets are the same, e.g.,

$$A^2 = A \times A, \qquad A^3 = A \times A \times A, \ \ldots$$

Power sets

Definition

The power set of A is the set of all subsets of A, denoted $\mathcal{P}(A)$. (Including both $\emptyset$ and A.)

Examples

1. $\mathcal{P}(\emptyset) = \{\emptyset\}$
2. $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$
3. $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

Do you see a pattern for how big $\mathcal{P}(A)$ will be if $|A| = n < \infty$?

How would you go about proving this?

Summation notation

Addition is a binary operation that is associative, which means that parentheses are permitted anywhere but required nowhere.

As such, we may write

$$((a_1 + a_2) + a_3) + a_4 = (a_1 + a_2) + (a_3 + a_4) = a_1 + a_2 + a_3 + a_4 = \sum_{k=1}^{4} a_k,$$

and the last term is called summation notation.

A finite series is an expression such as $a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$. We say:

- the variable k is the index
- the expression a_k is the general term of the series
- the values below and above the summation symbol are the initial index and terminal index, respectively.

Another associative binary operation is multiplication. The product of elements $a_1, \ldots, a_n$ is written in product notation, using a Π instead of a Σ:

$$a_1 a_2 \cdots a_n = \prod_{k=1}^{n} a_k.$$

Associative set operations

Let $A_1, A_2, \ldots, A_n$ be sets. Then:

(a) $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

(b) $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

(c) $A_1 \times A_2 \times \cdots \times A_n = \bigtimes_{i=1}^{n} A_i$

(d) $A_1 \oplus A_2 \oplus \cdots \oplus A_n = \bigoplus_{i=1}^{n} A_i$.

Examples

For $A_1 = \{0, 2, 3\}$, $A_2 = \{1, 2, 3, 6\}$, $A_3 = \{-1, 0, 3, 9\}$,

(a) $\bigcap_{i=1}^{3} =$

(b) $\bigcup_{i=1}^{3} =$

(c) $\bigtimes_{i=1}^{3} =$

(d) $\bigoplus_{i=1}^{3} =$

Distributive laws

See if you can find a general fomula for the following two expressions by looking at the cases where $n = 2$ and drawing a Venn diagram:

$$\boldsymbol{A} \cap \left(\bigcup_{i=1}^{n} B_i \right) =$$

$$\boldsymbol{A} \cup \left(\bigcap_{i=1}^{n} B_i \right) =$$