Lecture 2.3: Equivalence and implication

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Equivalence

Equivalence is to logic as equality is to algebra.

Definition

Propositions r and s generated by S are equivalent if and only if $r \leftrightarrow s$ is a tautology. We denote this as $r \Leftrightarrow s$.

Examples

The following are all equivalences:

1. $(p \wedge q) \vee (\neg p \wedge q) \Leftrightarrow q$
2. $p \to q \Leftrightarrow \neg q \to \neg p$
3. $p \vee q \Leftrightarrow q \vee p$
4. Every tautology is equivalent to 1
5. Every contradiction is equivalent to 0.

Equivalence

Though it's convenient to use operations such as $\vee$, $\rightarrow$, and $\leftrightarrow$, they are technically unnecessary.

All of them can be achieved using just $\wedge$ and $\neg$:

p	q	$p \vee q$	$\neg(\neg p \wedge \neg q)$	$p \rightarrow q$	$\neg p \vee q$	$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
0	0	0	0	1	1	1	1
0	1	1	1	1	1	0	0
1	0	1	1	0	0	0	0
1	1	1	1	1	1	1	1

Perhaps surprisingly, we can actually generate *every* Boolean function using just one operation. Either of the following will suffice:

p	q	$p \mid q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

The symbol $\mid$ is called a Sheffer stroke, and $\downarrow$ is called a Pierce arrow.

A universal operation

The function $p \mid q$ is also called NAND (NOT-AND), because it is equivalent to $\neg(p \wedge q)$.

Similarly, the function $p \downarrow q$ is also called NOR (NOT-OR), because it is equivalent to $\neg(p \vee q)$.

p	q	$p \wedge q$	$p \mid q$	$p \vee q$	$p \downarrow q$
0	0	0	1	0	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	0	1	0

Remark

The NAND and NOR operations are said to be universal because either one of them can generate *all* Boolean functions:

- $\neg p \Leftrightarrow p \mid p \Leftrightarrow p \downarrow p$
- $p \wedge q \Leftrightarrow (p \mid q) \mid (p \mid q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$
- $p \vee q \Leftrightarrow (p \mid p) \mid (q \mid q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$.

Basic logical equivalences

Most logical equivalence can be deduced by applying a few basic logic laws.

Each of the following laws has a dual law obtained by exchanging the symbols:

- $\wedge$ with $\vee$
- 0 with 1.

Basic law	**Name**	**Dual law**
$p \vee q \Leftrightarrow q \vee p$	Commutativity	$p \wedge q \Leftrightarrow q \wedge p$
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associativity	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributivity	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \vee 0 \Leftrightarrow p$	Identity	$p \wedge 1 \Leftrightarrow p$
$p \wedge \neg p \Leftrightarrow 0$	Negation	$p \vee \neg p \Leftrightarrow 1$
$p \vee p \Leftrightarrow p$	Idempotent	$p \wedge p \Leftrightarrow p$
$p \wedge 0 \Leftrightarrow 0$	Null	$p \vee 1 \Leftrightarrow 1$
$p \wedge (p \vee q) \Leftrightarrow p$	Absorption	$p \vee (p \wedge q) \Leftrightarrow p$
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	DeMorgan's	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Implication

Suppose you are told that one of the two propositions is true, and the other is false:

x: *The money is behind Door A*

y: *The money is behind Door A or Door B*

Which door would you choose?

Definition

Let S be a set of propositions and let r, s be generated by S. We say that r implies s if $r \rightarrow s$ is a tautology. We denote this as $r \Rightarrow s$.

Example

The implication $p \Rightarrow (p \vee q)$ is called disjunction addition.

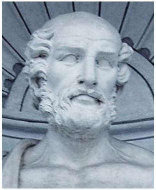
p	q	$p \rightarrow q$	$p \vee q$	$p \rightarrow p \vee q$
0	0	1	0	1
0	1	1	1	1
1	0	0	1	1
1	1	1	1	1

Basic logical implications

Name	Implication
Detachment	$(p \to q) \wedge p \Rightarrow q$
Indirect reasoning	$(p \to q) \wedge \neg q \Rightarrow \neg p$
Disjunctive addition	$p \Rightarrow (p \vee q)$
Conjuctive simplification	$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$
Disjunctive simplification	$(p \vee q) \wedge \neg p \Rightarrow q$ and $(p \vee q) \wedge \neg q \Rightarrow p$

The "detachment" law is often called *modus ponens*, short for *modus ponendo ponens* (Latin for "mode that affirms by affirming").

It dates back to the ancient Greek philosopher Theophrastus (371–287 BC), who succeeded the founder Aristotle of the Peripatetic school.



The "indirect reasoning" law is often called *modus tollens*, short for *modus tollendo tollens* (Latin for "mode that denies by denying").

Equivalences involving the conditional operator

Name	Equivalence
Transitivity	$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$
Conditional equivalence	$(p \to q) \Leftrightarrow \neg p \vee q$
Biconditional equivalence	$(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
Contrapositive	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$

The importance of language

Sometimes language can be misleading. And other times, phrasing a logical statement differently can make it less or more clear.

Example 1

Consider the following two statements:

(i) *If Alice or Barb go, then Chris will go.*

(ii) *If Alice will go, Chris will go, or if Barb will go, then Chris will go.*

Are these equivalent? If not, does one imply the other?

Example 2

Consider the following two statements:

(i) *If you get 90% in this class, or an A on the final, then you'll get an A.*

(ii) *I'm planning on putting at least one of the following statements on the syllabus:*

- *"If you get 90% in the class, you'll get an A"*
- *"If you get an A on the final, you'll get an A"*

Are these equivalent? If not, does one imply the other?

Summary

In terms of truth tables...

- Tautology means "*the truth table has all* 1*s*"
- Contradiction means "*the truth table has all* 0*s*"
- Equivalence $p \Leftrightarrow q$ means "*has the same truth table*"
- Implication $p \Rightarrow q$ means "*all the 1s in p's truth table are also in q's*"

The NAND and NOR operations are universal because they can generate any Boolean operation.

This means that every digital circuit can be constructed only with using NAND gates (or only NOR gates).