Lecture 3.6: Quotient, remainder, ceiling and floor

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Math 4190, Discrete Mathematical Structures

Division and remainder

Theorem

Given any $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, there exists unique integers q, r such that

$$n = dq + r, \qquad\qquad 0 \leq r < d.$$

We call $q := n \operatorname{div} d$ and $r := n \operatorname{mod} d$ the quotient and remainder, respectively.

Examples

1. Compute $365 \operatorname{div} 7$ and $356 \mod 7$.
2. Suppose $m \mod 11 = 6$. Compute $4n \mod 11$.
3. Given $n \in \mathbb{Z}$, compute $n^2 \mod 4$.

Division and remainder

If $n \in \mathbb{Z}$ is odd, then $n^2 \bmod 8 = 1$. Equivalently,

$$\forall \textit{ odd } n, \exists m \in \mathbb{Z} \textit{ such that } n^2 = 8m + 1.$$

Ceiling and floor

Definition

Given $x \in \mathbb{R}$, the floor of x is defined as

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \leq x < n+1.$$

The ceiling of x is defined as

$$\lceil x \rceil = n \quad \Leftrightarrow \quad n-1 < x \leq n.$$

Questions

Are the following true or false?

1. $\lfloor x-1 \rfloor = \lfloor x \rfloor - 1$
2. $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.

Ceiling and floor

Proposition

For all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

Proof

By definition, $n \leq x < n + 1$, where $\lfloor x \rfloor = n$.

Adding m yields

$$\underbrace{n + m}_{=\lfloor x+m \rfloor} \leq x + m < n + m + 1.$$

But $\lfloor x \rfloor = n$ implies that $n + m = \lfloor x \rfloor + m$. $\square$

Ceiling and floor

Proposition

For all integers n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \dfrac{n}{2} & n \text{ is even} \\ \dfrac{n-1}{2} & n \text{ is odd.} \end{cases}$$

Ceiling and floor

Proposition

For all integers n and d,

$$n \text{ div } d = \left\lfloor \frac{n}{d} \right\rfloor, \qquad \text{and} \qquad n \text{ mod } d = n - d \left\lfloor \frac{n}{d} \right\rfloor.$$