Lecture 4.1: Binary relations on a set

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Math 4190, Discrete Mathematical Structures

Motivation

We know what it means for one number to be less than (or equal to) another.

We know what it means for two numbers to be equal.

In this lecture, we will generalize these concepts to other sets.

We will do this by defining the notion of a binary relation on a set. Two special cases:

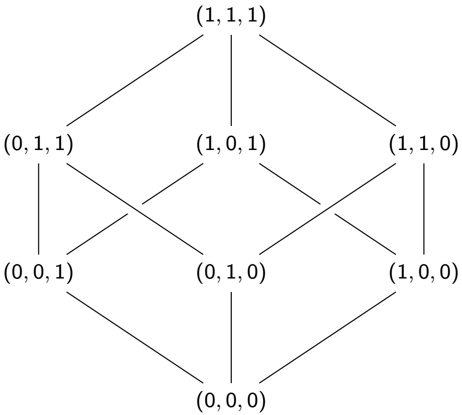
- partial orders (often written $\leq$, $\subseteq$, $\preceq$, etc.)
- equialence relations (often written $\equiv$, $\cong$, $\sim$, etc.)

Let's start with some visual examples to motivate the concepts that follow.

A "partial order": the Boolean lattice

Consider the set of length-3 binary vectors (or strings).

The following Hasse diagram shows what it means for one string to be "less than" another.



This is an example of a partially ordered set. Note that some strings are incomparable.

Another "partial order": partitions of $\{1, 2, 3, 4\}$

Say that a partition π is "less than" π' if π is a refinement of π'.

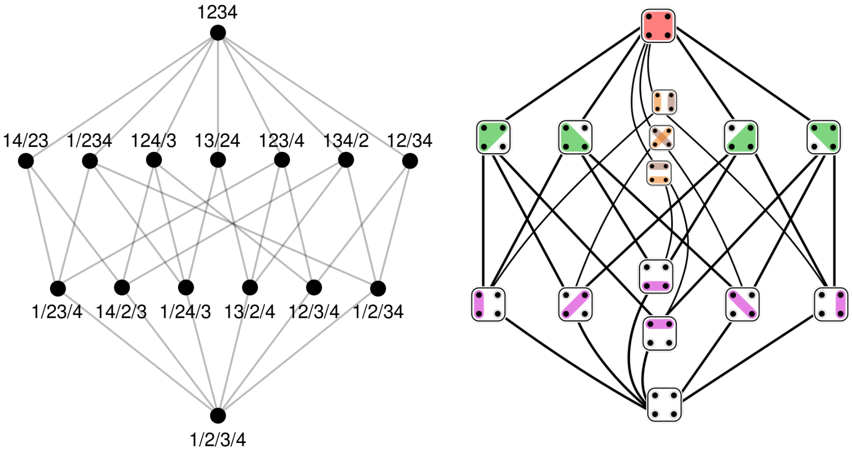


Figure: Two drawing of the partition lattice for $n = 4$.

Examples of "equivalence relations"

Example 1: isomorphic graphs

Let S be the following set of graphs with vertex set $V = \{1, 2, 3, 4\}$. Two graphs G_1, G_2 are isomorphic if they "have the same structure", denoted $G_1 \cong G_2$.

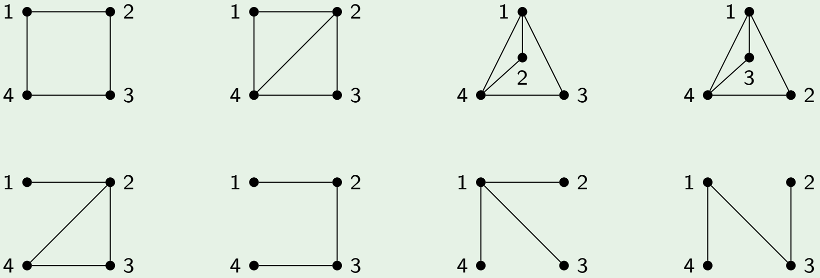


Figure: Some graphs on 4 vertices.

Example 2: similar matrices

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with coefficients from $\mathbb{C}$.

Two matrices A, B are similar if $A = PBP^{-1}$, for some matrix P.

Binary relations

Definition

Let A and B be sets. A (binary) relation from A into B is any subset R of $A \times B$.

If $A = B$ (usually the case), we say that R is a relation on A.

Examples

Let $A = \{1,2,3\}$ and $B = \{4,5\}$. Then $R = \{(1,4),(2,4),(3,5)\}$.

There are several common ways to express that a is related to b:

- $(a,b) \in R$,
- $a \; R \; b$,
- Define a symbol, e.g., $a \preceq b$ or $a \sim b$.

Most binary relations that we encounter are of the "less than" or "equivalence" type.

Common "less than" relations

- On $\mathbb{Z}$ (or $\mathbb{R}$, etc.): $a \leq b$, or $a < b$
- On 2^S for a fixed S: $A \subseteq B$, or $A \subsetneq B$
- On $\mathbb{Z}^+$: $a \mid b$
- On partitions: refinement

Common "equivalence" relations

- On $\mathbb{Z}$ (or $\mathbb{R}$, etc.) $a = b$
- On $\mathbb{Z}$: $a \equiv b \pmod{12}$
- On 2^S: $A \equiv B$ iff $|A| = |B|$
- On matrices: $A \cong B$ iff $A = PBP^{-1}$

Basic properties of binary relations

Definition

A relation R on a set A is:

(i) reflexive if $(a, a) \in R$ for all $a \in A$;

(ii) transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$;

(iii) symmetric if $(a, b) \in R \implies (b, a) \in R$;

(iv) anti-symmetric if $(a, b) \in R \implies (b, a) \notin R$ for all $a \neq b$.

Common "less than" relations

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- On 2^S for a fixed S: $A \subseteq B$
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Remark

The "less than" relations are antisymmetric. The "equivalence" relations are symmetric.

Both are reflexive and transitive.

Basic properties of binary relations

Examples

Let's determine whether the following relations are reflexive, transitive, symmetric, or antisymmetric:

1. $\leq$ on $\mathbb{R}$
2. $<$ on $\mathbb{R}$
3. $\subseteq$ on 2^S
4. $\subsetneq$ on 2^S
5. $\equiv_n$ on $\mathbb{Z}$
6. $\mid$ on $\mathbb{Z}^+ := \{1, 2, \dots\}$
7. $\mid$ on $\mathbb{N} := \{0, 1, 2, \dots\}$
8. $\mid$ on $\mathbb{Z}$
9. similarity on the set of $n \times n$ matrices
10. $R = \{(1,1), (1,2), (3,3), (3,4), (4,3), (4,4)\}$ on $\{1, 2, 3, 4\}$.

The two most common types of binary relations

Definition

A partial order on a set P is a relation that is

(i) reflexive,

(ii) transitive,

(iii) antisymmetric.

We can denote this as $(P, \preceq)$, and call P a poset, for short.

Definition

An equivalence relation on a set A is a relation that is

(i) reflexive,

(ii) transitive,

(iii) symmetric.

We can always visualize a relation R on a finite set A with a directed graph (digraph):

- the vertex set is A;
- include a directed edge $a \to b$ if $(a, b) \in R$.

Note that the digraph of a partial order (excluding self-loops) will be acyclic, and the digraph of an equivalence relation will be bidirected.

Irreflexive relations

Definition

A relation R on a set A is irreflexive if $(a, a) \notin R$ for all $a \in A$.

Remark

Every partial order $\preceq$ on P has a related irreflexive relation $\prec$.

More remarks

- Irreflexive and non-reflexive are different concepts.
- Antisymmetric and non-symmetric are different concepts.

To see what the the opposite of a property is, take the negation. For example,

$$R \text{ is transitive} \;\Leftrightarrow\; \forall (a, b), (b, c) \in R,\; (a, c) \in R$$

To see what non-transitive means, take the negation:

$$\begin{aligned} R \text{ is non-transitive} &\Leftrightarrow \neg\big[\forall (a, b), (b, c) \in R,\; (a, c) \in R\big] \\ &\Leftrightarrow \exists (a, b), (b, c) \in R \text{ such that } (a, c) \notin R. \end{aligned}$$

n-ary relations

The relations we've seen are all binary relations. But we can define n-ary relations similarly.

Definition

Let $A_1, \ldots, A_n$ be sets. An n-ary relation is a subset R of $A_1 \times A_2 \times \cdots \times A_n$.

Clearly, binary relations are the special case of $n = 2$.

Higher-order binary relations arise in database management systems.

For example, suppose a hospital keeps a database of its patients stored in a table with 4 entries:

1. $A_1 =$ patient IDs (positive integers)
2. $A_2 =$ patient names (strings)
3. $A_3 =$ dates in MMDDYYYY format (positive integers)
4. $A_4 =$ reason for emergency room admission (strings)

This is a 4-ary relation on $A_1 \times A_2 \times A_3 \times A_4$, defined by

$$(a_1, a_2, a_3, a_4) \in R \quad \Leftrightarrow \quad \text{patient with ID number } a_1 \text{ and name } a_2 \text{ was admitted on date } a_3 \text{ for reason } a_4.$$

n-ary relations and database programming

Consider a database R which contains the following 4-tuples:

(120423, Alice Smith, 01302018, flu)

(093789, John Doe, 02092018, broken leg)

(839412, Alan Johnson, 08112018, chest pains)

(042185, Catherine Greenman, 11202018, pregnancy)

(290384, Maeve O'Neil, 11202018, appendicitis)

In the database language SQL, the reults of the query

```
SELECT Patient_ID#, Name FROM R WHERE
Admission_Date = 11202018
```

would be

042185	Catherine Greenman
290384	Maeve O'Neil

Mathematically, this is done by intersecting $A_1 \times A_2 \times \{11202018\} \times A_4$ with R, and then projecting onto the first two coordinates.