

Read: Lax, Chapter 2, pages 13–18.

1. Let K be a finite field. The *characteristic* of K , denoted $\text{char } K$, is the smallest positive integer n for which $n1 := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$.
 - (a) Prove that the characteristic of K is prime.
 - (b) Show that K is a vector space over \mathbb{Z}_p , where $p = \text{char } K$.
 - (c) Show that $|K|$ is a prime power.
 - (d) Show that if K and L are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then m divides n .
2. Let X be a vector space over a field K and let X' be the set of linear functions from X to K , also known as the *dual space* of X .
 - (a) Let x_1, \dots, x_n be a basis for X . Define $\ell_j \in X'$ by $\ell_j(x_i) = \delta_{ij}$. Show that ℓ_1, \dots, ℓ_n is a basis for X' ; it is called the *dual basis* of x_1, \dots, x_n .
 - (b) Find the dual basis of $x_1 = (1, -1, 3)$, $x_2 = (0, 1, -1)$, and $x_3 = (0, 3, -2)$ in $X = \mathbb{R}^3$.
 - (c) Express the scalar function $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, ℓ_1, ℓ_2, ℓ_3 , from Part (b).

3. Let S be a subset of X . The *annihilator* of S is the set

$$S^\perp = \{\ell \in X' \mid \ell(s) = 0 \text{ for all } s \in S\}.$$

- (a) Show that $\text{span}(S)$ is the intersection of all subspaces T_α of X that contain S :

$$\text{span}(S) = \bigcap_{S \subseteq T_\alpha \subseteq X} T_\alpha,$$

making it well-founded to speak of the “*smallest subspace of X that contains S* .”

- (b) Show that S^\perp is a subspace of X' , and that $S^\perp = \text{span}(S)^\perp$.

4. Let \mathcal{P}_2 be the vector space of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ over \mathbb{R} , with degree ≤ 2 . Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j: \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad \ell_j(p) = p(\xi_j) \quad \text{for } j = 1, 2, 3.$$

- (a) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space \mathcal{P}'_2 .
- (b) Find polynomials $p_1(x), p_2(x), p_3(x)$ in \mathcal{P}_2 of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in \mathcal{P}'_2 .

5. Let W be the subspace of \mathbb{R}^4 spanned by $(1, 0, -1, 2)$ and $(2, 3, 1, 1)$. Which linear functions $\ell(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W ? Write your answer by giving an explicit basis of W^\perp .