

Read: Lax, Chapter 3, pages 19–25, 28–31.

1. Let $T: X \rightarrow U$ be a linear map. Prove the following:
 - (a) The image of a subspace of X is a subspace of U .
 - (b) The inverse image of a subspace of U is a subspace of X .

2. Let X and U be vector spaces, and suppose that Y is a subspace of X . Let $Q: X \rightarrow X/Y$ be the canonical quotient map sending $x \mapsto \{x\}$, and let $T: X \rightarrow U$ be a linear map. State and prove necessary and sufficient conditions for the existence of a unique linear map $S: X/Y \rightarrow U$ such that $T = S \circ Q$. When this happens, the map T is said to *factor through* the quotient space, as shown by the following commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{T} & U \\
 \searrow Q & & \nearrow S \\
 & X/Y &
 \end{array}$$

3. Suppose $T: X \rightarrow X$ is a linear map of rank 1.
 - (a) Show that there exists $c \in K$ such that $T^2 = cT$.
 - (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

4. Suppose that $S, T: X \rightarrow X$ are linear maps.
 - (a) Show that $\text{rank}(S + T) \leq \text{rank}(S) + \text{rank}(T)$.
 - (b) Show that $\text{rank}(ST) \leq \text{rank}(S)$.
 - (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

For each of these, without using matrices, give an explicit example of where equality holds, and another where it does not.

5. Let $T: X \rightarrow X$ be linear, with $\dim X = n$.
 - (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.
 - (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.
 - (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.
 - (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.
 - (e) Show there exists a linear map $S: X \rightarrow X$ such that $ST = TS$ and $ST^{n+1} = T^n$.