Read: Lax, Chapter 5, pages 55–56, and Appendix 4, pages 313–316.

1. Prove the following properties of the trace function:

   (a) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $m \times n$ matrices A and $n \times m$ matrices B.

   (b) If A is square, write down a formula for $\operatorname{tr}(AA^T)$ in terms of a_{ij}.

2. Let U, V, and X be vector spaces over a field K. Define a map

$$\tau \colon U \times V \longrightarrow U \otimes V\,, \qquad \tau(u,v) = u \otimes v\,.$$

   (a) Prove that τ is bilinear.

   (b) Prove that for any linear map $L \colon U \otimes V \to X$, the mapping $\beta := L \circ \tau$ is bilinear.

   (c) Prove the universal property of tensor products: for any bilinear map $\beta \colon U \times V \to X$, there is a unique linear mapping $L \colon U \otimes V \to X$ such that $\beta = L \circ \tau$:

$$\begin{tikzcd} U \times V \arrow[rr, "\beta"] \arrow[dr, "\tau"'] & & X \\ & U \otimes V \arrow[ur, dashed, "L"'] & \end{tikzcd}$$

3. If $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ are bases for U and V, then the pure tensors $\{u_i \otimes v_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ clearly span $U \otimes V$. Show that these are linearly independent, and conclude that $\dim(U \otimes V) = (\dim U)(\dim V)$. [*Hint*: Apply the universal property to the canonical basis $\{f_{ij}\}$ of bilinear functions $U \times V \to K$.]

4. Use the universal property of the tensor product to prove the following results:

   (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);

   (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;

   (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.

5. Let W be a vector space with basis $\{w_{ij} \mid 1 \le i \le n,\ 1 \le j \le m\}$. Define the linear map

$$\alpha \colon W \longrightarrow U \otimes V, \qquad \alpha \colon w_{ij} \longmapsto u_i \otimes v_j.$$

   To show this is an isomorphism, we would like to define the (inverse) map $\beta \colon U \otimes V \to W$. But to do so, we would need a basis for $U \otimes V$. So instead, define a map

$$\tilde{\beta} \colon F_{U \times V} \longrightarrow W, \qquad \tilde{\beta} \colon e_{\Sigma a_i u_i, \Sigma b_j v_j} \longmapsto \sum_{i,j} a_i b_j w_{ij}.$$

   where $F_{U \times V}$ is the *free vector space* with basis $U \times V$:

$$F_{U \times V} = \Big\{ \sum c_{uv} e_{u,v} \quad \mid \quad u \in U,\ \ v \in V \Big\}.$$

   (a) Show that $N_q \subseteq N_{\tilde{\beta}}$, where $q \colon F_{U \times V} \to U \otimes V$ is the canonical quotient, and apply the universal property of quotient maps (see HW 3) to get a map $\beta \colon U \otimes V \to W$.

   (b) Show that α and β are inverses by verifying that $\alpha \circ \beta = \mathrm{Id}_{U \otimes V}$ and $\beta \circ \alpha = \mathrm{Id}_W$.