Read: Lax, Chapter 7, pages 77–100.

1. Consider the vector space of all polynomials in $\mathbb{C}[x, y]$ of total degree at most 2,

$$X = \Big\{ \sum a_{i,j} x^i y^j \mid a_{i,j} \in \mathbb{C},\ 0 \leq i + j \leq 2 \Big\},$$

and consider the linear map

$$D : X \longrightarrow X, \qquad f \longmapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

   (a) Write D in matrix form, with respect to the ordered basis $1, x, y, x^2, xy, y^2$.

   (b) Find the minimal and characteristic polynomials, and the Jordan canonical form.

   (c) Find a basis of generalized eigenvectors of D.

   (d) Conjecture how this generalizes to polynomial of total degree at most n.

2. Let X be the xy-plane and $A\colon X \to X$ be a $45°$ counterclockwise rotation.

   (a) Let $v_0 = e_1 = (1, 0)^T$, $v_1 = Av_0$, and $v_2 = A^2 v_0$. Write v_2 as a linear combination of v_0 and v_1, and use this to find the minimal polynomial of A.

   (b) Write the matrix of A with respect to the basis v_0, v_1, and compare it to the Jordan canonical form.

   (c) Repeat the previous parts for a linear map $A\colon X \to X$ with eigenvalues $\lambda_{1,2} = re^{\pm i\theta}$.

   (d) Re-write the matrices in Part (c) in terms of a and b, where $a \pm bi = re^{\pm i\theta}$.

3. Consider the following matrix over $\mathbb{R}$:

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

   (a) Show that if a $\deg f(x) < n$, then $f(M) \neq 0$. [*Hint*: Show that $f(M)e_1 \neq 0$.]

   (b) Show that the minimal polynomial of M is $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

4. Let X be a vector space over $\mathbb{R}$ with basis $\{x_1, x_2, x_3, x_4\}$ and let $T : X \to X$ be a linear map such that

$$T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.$$

Find the rational and Jordan canonical forms of T. Is T diagonalizable over $\mathbb{C}$?