Read: Lax, Chapter 7, pages 77–100.

1. Let $X = \mathbb{R}^3$, and define the inner product by

$$\langle x, y\rangle = y^T A x = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Find the norm of the three unit basis vectors e_1, e_2, and e_3, the angles between them, and the orthogonal complements of the lines that they span.

2. Given a linear map $A\colon X \to X$, define $f\colon X \to X$ by $f(x, y) = x^T A y$.

   (a) Write the inner product $f(x, y) = 3x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - x_2y_3 - x_3y_2 + 3x_3y_3$ as $f(x, y) = x^T A y$.

   (b) Find an orthonormal basis v_1, v_2, v_3 of $\mathbb{R}^3$ so that with respect to this basis, $f(z, w) = z^T D w$ for some diagonal matrix D.

   (c) Write a formula for $f(z, w)$ like in Part (b), but with respect to this new basis.

   (d) State and prove necessary and suffcient conditions on A for f to be an inner product.

3. Let Y, Z be subspaces of an inner product space X.

   (a) Show that $Y \subseteq Y^{\perp\perp}$, with equality holding if $\dim X < \infty$.

   (b) Give an example of an infinite dimensinal space where equality does not hold.

   (c) Show that $(Y + Z)^\perp = Y^\perp \cap Z^\perp$.

4. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1, 2, 1, 1)$, $y_2 = (1, -1, 0, 2)$ and $y_3 = (2, 0, 1, 1)$.

   Then write the vector $v = (4, 1, 2, 4)$ in this basis.

5. Let X be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on X by

$$(f, g) = \int_0^1 f(t)g(t)\, dt\,.$$

   Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$.

   (a) Use the Gram-Schmidt process to construct an orthonormal basis for Y.

   (b) Write $f(x) = 2x^3 - x^2 + 4$ using your basis from Part (a).